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THE NUMERICAL SOLUTION OF A SINGULAR PERTURBATION PROBLEM IN THE DOMAIN EXTERIOR OF A CIRCLE

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ABSTRACT

The numerical solution of a singular perturbation problem in the exterior domain of a circle is investigated.

First the boundary value problem is solved analytically and the numerical properties of this explicit expression are examined. Thereafter asymptotic expansions for the solution are developed along the lines given by FRIEDLANDER [1954] and WAECHTER [1968]. Based on the accumulated information, numerical approximations to the solution are obtained.

1. THE ANALYTICAL SOLUTION

In this report we consider the function $u(x,y;\varepsilon)$, a function of two variables x and y, containing a small parameter ε, which is the solution of the partial differential equation

$$\varepsilon \Delta u = \frac{\partial}{\partial x} u \tag{1.1}$$

in the region $D = \{(x,y) \mid x^2 + y^2 \geq 1\}$ in $\mathbb{R}^2$ and which satisfies the boundary conditions

$$u(x,y;\varepsilon) = 1 \qquad \text{for } x^2 + y^2 = 1 \tag{1.2a}$$

and

$$u(x,y;\varepsilon) \to 0 \qquad \text{for } x^2 + y^2 \to \infty. \tag{1.2b}$$

The boundary conditions suggest a treatment in polar coordinates. The asymmetry with respect to polar coordinates lies in the right-hand side of (1.1) only. This asymmetry is removed and a simpler form is obtained by introducing the function

$$v(x,y;M) = e^{-Mx}\, u(x,y;\tfrac{1}{2M}) \ , \tag{1.3}$$

where $M = \frac{1}{2\varepsilon}$ is a large parameter. Passing to polar coordinates yields

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = M^2 v \tag{1.4}$$

with the boundary conditions

$$v(1,\theta) = e^{-M\cos\theta} = e^{M\cos(\pi-\theta)} \tag{1.5a}$$

and

$$v(\infty,\theta) = 0 \ . \tag{1.5b}$$

Using the technique of separation of variables, we write $v(r,\theta) =$ $= f(\omega,r)\, g(\omega,\theta)$. This yields

(1.6a) $g_{\theta\theta} = -\omega^2 g$

and

(1.6b) $r^2 f_{rr} + r f_r - \{\omega^2 + M^2 r^2\} f = 0.$

The solution of equation (1.6b) is a modified Besselfunction $I_\omega(Mr)$ or $K_\omega(Mr)$. Only the function $K_\omega(Mr)$ will satisfy the boundary condition at infinity (1.5b). Hence the solution of (1.4) can be written in the general form

(1.7) $$\int_c A(\omega) K_\omega(Mr) e^{i\omega\theta}\, d\omega.$$

In order to satisfy the boundary conditions (1.5), the path of integration of (1.7) and the weight function $A(\omega)$ have to be chosen such that

$$\int_c A(\omega) K_\omega(M) e^{i\omega\theta}\, d\omega = e^{-M\cos\theta}.$$

At the end of section 3, we will find an asymptotic expansion for (1.7)

(1.8) $$w(r,\theta;M) \sim \int_c \frac{I_\nu(M)}{K_\nu(M)} K_\nu(Mr) e^{i\nu\theta}\, d\nu \qquad \text{for } M \to \infty$$

which, in addition, satisfies $w(1,\theta;M) = e^{-M\cos\theta}$. Consequently the following asymptotic relation will hold

(1.9) $$w(r,\theta;M) \sim v(r,\theta;M) \qquad \text{for } M \to \infty.$$

As a special case of (1.7) we consider

(1.10) $$v(r,\theta;M) = \sum_{n=-\infty}^{+\infty} A_n K_n(Mr) e^{\pm in\theta}.$$

Now A_n has to be chosen such that boundary condition (1.5a) is satisfied. In order to fit this condition we use the "generating function" of $I_n(x)$

$$e^{x(t+1/t)/2} = \sum_{n=-\infty}^{+\infty} I_n(x) t^n.$$

Here we take $t = e^{i\theta}$ and $x = -M$, whence, for all M and θ, the following relations hold

$$e^{-M\cos\theta} = \sum_{n=-\infty}^{+\infty} I_n(-M)e^{in\theta} = \sum_{n=-\infty}^{+\infty} (-1)^n I_n(M)e^{in\theta}$$

$$v(1,\theta;M) = \sum_{n=-\infty}^{+\infty} A_n K_n(M)e^{in\theta}.$$

From this it directly follows that

$$(1.11) \qquad v(r,\theta;M) = \sum_{n=-\infty}^{+\infty} \frac{(-)^n I_n(M) K_n(Mr)}{K_n(M)} e^{in\theta} =$$

$$= \sum_{n=-\infty}^{+\infty} \frac{I_n(M) K_n(Mr)}{K_n(M)} e^{in(\pi-\theta)} =$$

$$= \frac{I_0(M) K_0(Mr)}{K_0(M)} + 2 \sum_{n=1}^{\infty} \frac{I_n(M) K_n(Mr)}{K_n(M)} \cos(n(\pi-\theta)).$$

Hence, the solution of the boundary value problem (1.1), (1.2) reads

$$(1.12) \qquad u(r,\theta,1/2M) = e^{Mr\cos\theta} \sum_{n=-\infty}^{+\infty} \frac{I_n(M) K_n(Mr)}{K_n(M)} e^{in(\pi-\theta)} =$$

$$= e^{Mr\cos\theta} \left[\frac{I_0(M) K_0(Mr)}{K_0(M)} + 2\sum_{n=1}^{\infty} \frac{I_n(M) K_n(Mr)}{K_n(M)} \cos(n(\pi-\theta))\right].$$

2. NUMERICAL CONSIDERATIONS ON THE EXPLICIT FORM OF THE SOLUTION

In order to obtain numerical approximations to $u(x,y;\varepsilon)$ based on the explicit form of the solution (1.11), it is important to consider not only the convergence of the infinite series but also the loss of significant digits during the summation. For, when ε takes values that are not too small (say $\varepsilon>0.1$), the expression (1.11) enables us to compute u with a reasonable accuracy over the whole domain of definition. For instance, for $\varepsilon = 0.1$ we obtain an accuracy of 6 digits when an algorithm for the computation of the Besselfunctions is used, which delivers a result that is accurate in 10 digits. However, for smaller values of ε the cancellation of significant digits impedes the computation. As an example we show computed values for u and an estimate of its relative error, due to cancellation, for some different values of ε.

y \ x	-2	-1	0	1	2	3	4
2	1.345(-7)	1.635(-4)	1.291(-2)	7.489(-2)	1.422(-1)	1.906(-1)	2.248(-1)
1	5.455(-6)	2.911(-2)	1.000	7.827(-1)	7.071(-1)	6.704(-1)	6.455(-1)
0	2.727(-5)	1.000	xxxxx	1.000	9.752(-1)	9.268(-1)	8.757(-1)

Values of u(x,y;0.1)

y \ x	-2	-1	0	1	2	3	4
2	3.771(-14)	4.987(-8)	3.007(-4)	1.062(-2)	4.013(-2)	7.412(-2)	1.048(-1)
1	5.282(-11)	1.137(-3)	1.000	7.559(-1)	6.860(-1)	6.534(-1)	6.334(-1)
0	1.216(-9)	1.000	xxxxx	1.000	9.956(-1)	9.774(-1)	9.500(-1)

Values of u(x,y;0.05)

Table 1. Numerical values of $u(x,y;\varepsilon)$, for $\varepsilon = 0.1, 0.05$, computed by evaluation of expression (1.12)

Table 2 shows that, due to cancellation of digits, the relative error increases and expression (1.12) can not be used for $\varepsilon < 0.05$.

y \ x	-2	-1	0	1	2	3	4
2	4.2(-13)	1.3(-13)	9.2(-12)	6.8(-11)	2.4(-10)	4.7(-10)	6.9(-10)
1	1.8(-13)	5.5(-13)	2.9(-11)	5.4(-10)	1.1(-9)	1.3(-9)	1.4(-9)
0	1 (-13)	1 (-13)		4.4(-9)	2.7(-9)	2.2(-9)	2.0(-9)

Relative error in u(x,y;0.1)

y \ x	-2	-1	0	1	2	3	4
2	1.2(-12)	9.3(-12)	3.9(-10)	2.1(-8)	2.9(-7)	1.3(-6)	3.0(-6)
1	3.0(-13)	2.0(-12)	4.4(-9)	1.5(-6)	7.3(-6)	1.3(-5)	1.7(-5)
0	1 (-13)	1 (-13)		9.7(-5)	5.8(-5)	4.6(-5)	4.0(-5)

Relative error in u(x,y;0.05)

y \ x	-2	-1	0	1	2	3	4
2	2.1(-12)	2.7(-9)	2.7(-5)	6.0(-1)	>0.5	>0.5	>0.5
1	1.0(-12)	6.6(-11)	1.4(-2)	>0.5	>0.5	>0.5	>0.5
0	1 (-13)	1 (-13)		0.5	>0.5	>0.5	>0.5

Relative error in u(x,y;0.02)

Table 2. Relative error, due to cancellation, in the numerical value of $u(x,y;\varepsilon)$, for $\varepsilon = 0.1, 0.05, 0.02$, when computed by evaluation of expression (1.12). The Besselfunctions are computed with a relative accuracy of 10^{-13}.

We will consider this effect in more detail. For large values of $M = 1/2\varepsilon$ we split off the asymptotic factor of the Besselfunctions. We define $I_n^*(M)$ by

$$(2.1) \qquad I_n(M) = \exp(M)(2\pi M)^{-\frac{1}{2}} I_n^*(M)$$

$$(2.2) \qquad K_n(M) = \exp(-M)\left(\frac{\pi}{2M}\right)^{\frac{1}{2}} K_n^*(M).$$

We notice that for $M \to \infty$

$$I_n^*(M) \sim 1 \quad \text{and} \quad K_n^*(M) \sim 1.$$

With this notation we write (1.1a) as

$$u(x,y;1/2M) = \exp(M(2-r+x))(2\pi Mr)^{-\frac{1}{2}} \cdot \tag{2.3}$$

$$\cdot \left\{ \frac{I_0^*(M)\ K_0^*(Mr)}{K_0^*(M)} + 2 \sum_{n=1}^{\infty} \frac{I_n^*(M)\ K_n^*(Mr)}{K_n^*(M)} \cos(n(\pi-\theta)) \right\} =$$

$$= \exp(M(x+r))u(r,\pi;1/2M) - 4\exp(M(2-r+x))(2\pi Mr)^{-\frac{1}{2}} \cdot$$

$$\sum_{n=1}^{\infty} \frac{I_n^*(M)\ K_n^*(Mr)}{K_n^*(M)} \sin^2\left(\frac{n}{2}(\pi-\theta)\right).$$

Hence no cancellation will take place for $\theta = \pi$ and the cancellation will be most significant for $\theta \approx 0$.

In addition formula (2.3) gives an indication to the behavior of the solution for moderate values of ε. The factor $\exp(M(2-r+x))$ shows that the behavior is primarily characterized by the exponential curves

$$2 - r + x = c, \qquad c \le 2,$$

or

$$(2-c)^2 + 2(2-c)x = y^2. \tag{2.4}$$

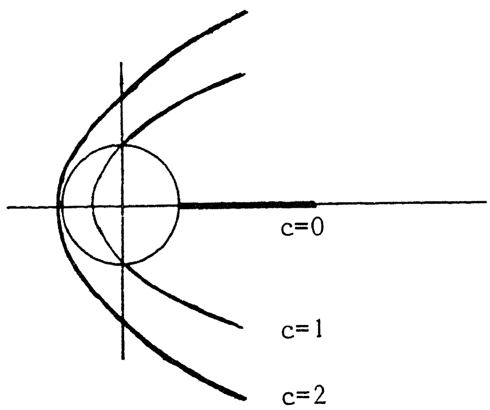


Figure 1

From expression (2.3) it also can be seen that, for large values of M, summation will fail in some part of the domain of definition, no matter what method is used.

Let us take, for instance, (x,y)=(1,0); then (2.3) reduces to

$$u(1,0) = e^{2M}(2\pi M)^{-\frac{1}{2}}\left\{I_0^*(M) + 2\sum_{k=1}^{\infty} (-1)^n I_n^*(M)\right\} .$$

The factor $e^{2M} M^{-\frac{1}{2}}$ strongly increases for $M \to \infty$, whereas $I_n^*(M) \to 1$ for all n. However, the sum

$$I_0^*(M) + 2\sum_{k=1}^{\infty} (-1)^n I_n^*(M)$$

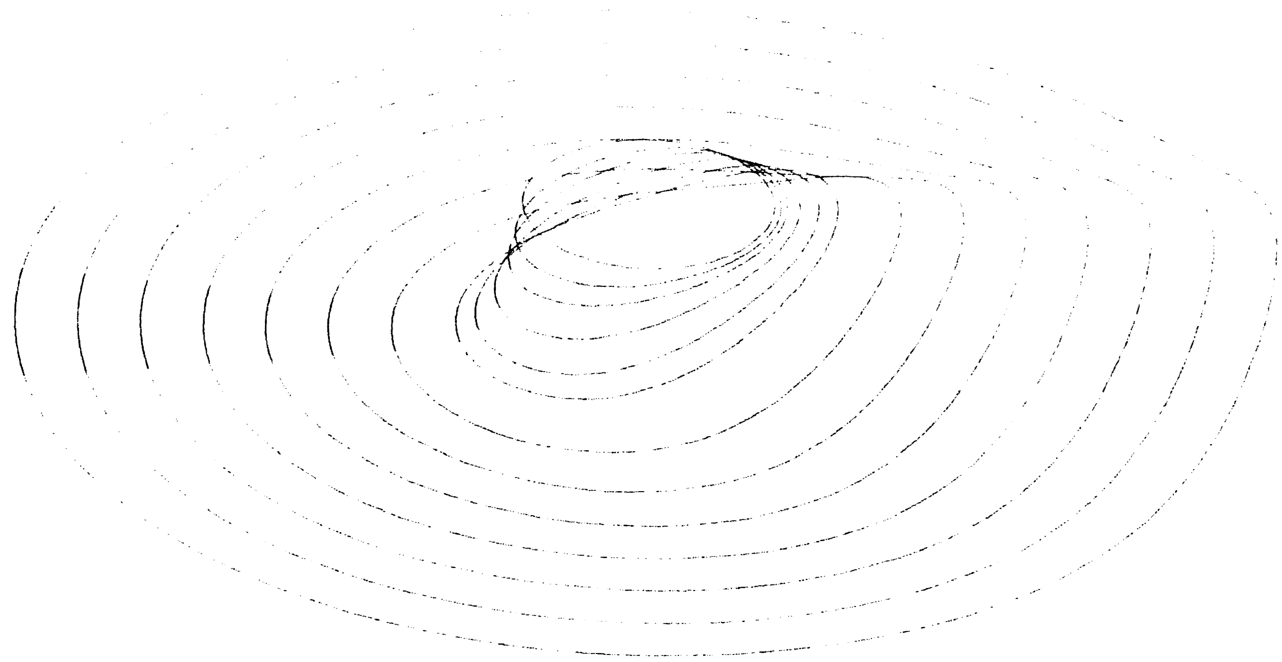
decreases as $e^{-2M} M^{\frac{1}{2}}$ for $M \to \infty$, since $u(1,0) = 1$. This shows that the summation given in (2.3) is unfit for the computation of u in a neighborhood of (1,0) for large values of M.

We conclude that the fitness of equation (1.12) for the computation of $u(x,y;\varepsilon)$ depends on the value of the parameter ε and also on the region where the solution is wanted. The neighborhood of (1,0) is the more improper region for straightforward summation and for large values of M the unfitness extends to the neighborhood of a domain which can roughly be characterized by $y^2 < 1+2x$.

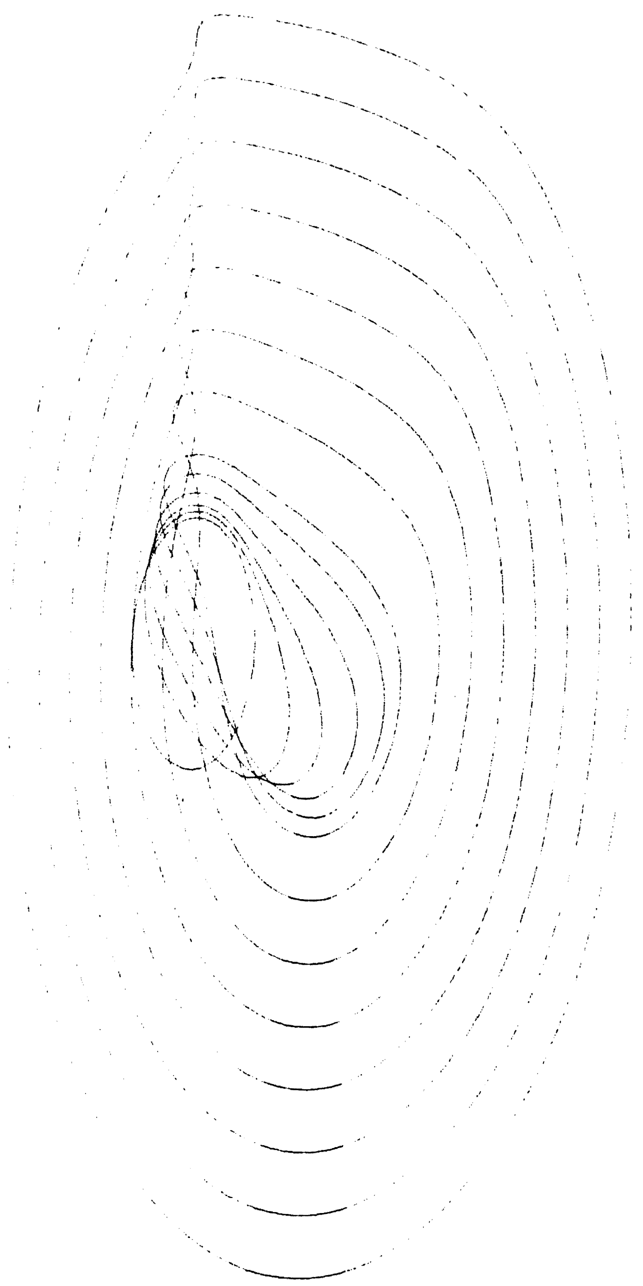
Figure 2. Parallel projections of cross-sections of the plane $z = u(x,y;\varepsilon)$ with the cylinders $x^2 + y^2 = r^2$ (r=1,1.05,1.1,1.2,1.5(.5)5) for ε = 0.5, 0.2, 0.055.

See next pages.

Figure 2a. $u(x,y;\varepsilon)$, $\varepsilon = 0.5$



<u>Figure 2b</u>. $u(x,y;\varepsilon)$, $\varepsilon = 0.2$.



<u>Figure 2c.</u> $u(x,y;\varepsilon)$, $\varepsilon = 0.055$.

3. COLLECTED RESULTS ON THE MODIFIED BESSELFUNCTIONS

In this section we summarize some results from ABRAMOWITZ & STEGUN [1964], FRIEDLANDER [1954,1958] and WAECHTER [1968]. In particular we shall use the results from FRIEDLANDER concerning the modified Besselfunction $K_\nu(z)$ for pure imaginary order. Asymptotic expansions are given for $K_{i\mu}(z)$, z and μ real and for the zeros of $K_{i\mu}(z)$.

3.1. The following properties (cf. FRIEDLANDER [1954]) of the modified Besselfunction $K_\nu(z)$ are relevant for the treatment given in the next section.

$$(3.1)\qquad K_{i\bar{\mu}}(z) = \overline{K_{i\mu}(z)} \quad , \quad K_{-\nu}(z) = K_\nu(z).$$

When s is real, the zeros of $K_{i\mu}(s)$ and $K'_{i\mu}(s)$ exist for real μ only. All zeros are simple and they satisfy the inequality

$$(3.2)\qquad |\mu_j| \geq s.$$

3.2. For z,μ real, $z,\mu > 0$ and $|\mu/2|$ large, the next asymptotic expansion holds for $\mu \to \infty$ (cf. FRIEDLANDER [1958])

$$(3.3)\qquad K_{i\mu}(z) \sim \sqrt{\frac{2\pi}{\mu}}\, e^{-\pi\mu/2}\left\{\sin(\mu\log\frac{2\mu}{ez} + \frac{\pi}{4}) + O(\frac{1}{\mu})\right\},$$

$$(3.4)\qquad K'_{i\mu}(z) \sim \frac{\sqrt{2\pi\mu}}{z}\, e^{-\pi\mu/2}\left\{\cos(\mu\log\frac{2\mu}{ez} + \frac{\pi}{4}) + O(\frac{1}{\mu})\right\}.$$

3.3. The following asymptotic expansion can be derived -by means of LANGERS' method- for real μ and s, μ and s both large and $\mu/s \sim 1$ (cf. FRIEDLANDER [1958])

$$(3.5)\qquad K_{i\mu}(s) \sim \sqrt{2}\,\pi\, e^{-\pi\mu/2}\left\{\left(\frac{\zeta}{s^2-\mu^2}\right)^{1/4} Ai(\zeta) + O(s^{-2/3})\right\},$$

where

$$(3.6)\quad \frac{2}{3}\,\zeta^{3/2} = (s^2-\mu^2)^{1/2} - \mu \arccos(\mu/s) \qquad \text{if } s \geq \mu$$

and

$$(3.7)\quad \frac{2}{3}(-\zeta)^{3/2} = \mu\,\mathrm{arccosh}(\mu/s) - (\mu^2-s^2)^{1/2} \qquad \text{if } s \leq \mu$$

$(0 \leq \arccos(\mu/s) \leq \pi/2)$.

3.4. For $\mu = s + \beta s^{1/3} + \mathcal{O}(s^{-1/3})$ and $s \to \infty$, it follows from section 3.3 that

$$(3.8)\quad K_{i\mu}(s) \sim \pi(\tfrac{2}{s})^{1/3}\, e^{-\pi\mu/2}\, \mathrm{Ai}(-2^{1/3}\beta) + \mathcal{O}(s^{-2/3}e^{\pi\mu/2})$$

$$(3.9)\quad K'_{i\mu}(s) \sim \pi(\tfrac{2}{s})^{2/3}\, e^{-\pi\mu/2}\, \mathrm{Ai}(-2^{1/3}\beta) + \mathcal{O}(s^{-1}e^{\pi\mu/2}).$$

Therefore we can give asymptotic expansions of the zeros of the functions $K_{i\mu}(s)$ and $K'_{i\mu}(s)$, considered as functions of μ, for $s \to \infty$

$$(3.10)\quad \mu_j(s) \sim s + \alpha_j(\tfrac{s}{2})^{1/3} + \mathcal{O}(s^{-1/3}),$$

where $-\alpha_j$ (Note: $\alpha_j>0$) are zeros of the Airyfunction $\mathrm{Ai}(\alpha)$ and $\mathrm{Ai}'(\alpha)$ respectively.

A more general asymptotic expansion than (3.8) can be given. WAECHTER [1968, appendix] mentions an asymptotic expansion of $K_{i\mu}(sr)$ for $s \to \infty$

$$sr = s + Rs^{1/3}, \qquad R = \mathcal{O}(1),$$
$$\mu = s + \beta s^{1/3} + \mathcal{O}(s^{-1/3})\ ;$$

this expansion reads

$$(3.11)\quad K_{i\mu}(sr) \sim \pi(\tfrac{2}{s})^{1/3}\, e^{-\pi\mu/2}\, \mathrm{Ai}(2^{1/3}(R-\beta)).$$

3.5. We mention also the asymptotic expansions for large orders (cf.

ABRAMOWITZ & STEGUN [1964])

(3.12) $$I_\nu(\nu z) = \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{1/4}} \left\{1 + \sum_{k=1}^{\infty} u_k(t)/\nu^k\right\}$$

(3.13) $$K_\nu(\nu z) = \sqrt{\frac{\pi}{2\nu}} \frac{e^{-\nu\eta}}{(1+z^2)^{1/4}} \left\{1 + \sum_{k=1}^{\infty} (-)^k u_k(t)/\nu^k\right\}$$

$$\eta = \sqrt{1 + z^2} + \log \frac{z}{1+\sqrt{1+z^2}} = \sqrt{1 + z^2} - \operatorname{arcsinh}(\tfrac{1}{z})$$

$$t = (1+z^2)^{1/4} \; ; \quad u_k(t) = \ldots.$$

These approximations hold uniformly with respect to z, $|\arg z| < \frac{\pi}{2} - \eta$, arbitrary $\eta > 0$, for $\nu \to \infty$.

3.6. For large $|z|$ we have the following asymptotic expansion of the Airy-function, $|\arg z| < \pi$ (cf. ABRAMOWITZ & STEGUN [1964])

(3.14) $$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} \frac{1}{z^{1/4}} e^{-\zeta}\{1 + O(\tfrac{1}{\zeta})\}$$

(3.15) $$\mathrm{Ai}'(z) \sim -\frac{1}{2\sqrt{\pi}} z^{1/4} e^{-\zeta}\{1 + O(\tfrac{1}{\zeta})\}$$

with $\zeta = \frac{2}{3} z^{3/2}$.

3.7. With the aid of the formulae (3.5) and (3.14) the next asymptotic expansion is easily verified for $r - 1 > O(s^{-2/3})$

(3.16) $$K_{i\mu}(sr) \sim \sqrt{\frac{\pi}{2s}} \frac{1}{(r^2-1)^{1/4}} \cdot$$

$$\cdot \exp\{-\pi\mu/2 - s[(r^2-1)^{1/2} - \arccos(\tfrac{1}{r})] + \beta s^{1/3} \arccos(\tfrac{1}{r})\}.$$

3.8. The following expansions directly follow from section 3.4.

$$(3.17)\quad \left[\frac{\partial}{\partial\mu} K_{i\mu}(s)\right]_{\mu=\mu_j} = -\pi\left(\frac{2}{s}\right)^{2/3} e^{-\pi\mu_j/2}\, Ai'(-\alpha_j)\ ,$$

$\mu_j = \mu_j(s)$ zero of $K_{i\mu}(s)$; α_j given by $\mu_j \sim s + \alpha_j\left(\frac{s}{2}\right)^{1/3}$.

$$(3.18)\quad \left[\frac{\partial}{\partial\mu} K'_{i\mu}(s)\right]_{\mu=\mu_j} = \frac{2\pi}{s}\alpha_j\, Ai(-\alpha_j) e^{-\pi\mu_j/2}\ ,$$

$\mu_j = \mu_j(s)$ zero of $K'_{i\mu}(s)$; α_j given by $\mu_j \sim s + \alpha_j\left(\frac{s}{2}\right)^{1/3}$.

4. ASYMPTOTIC EXPANSIONS OF THE SOLUTION $u(x,y;\varepsilon)$

4.1. Another analytic expression for the solution

First we try to find an asymptotic solution derived from the analytic solution. According to WAECHTER [1968], we apply the summation formula of Poisson

$$(4.1)\qquad \sum_{k=-\infty}^{+\infty} F(k) = \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2\pi imx} F(x)dx$$

to the analytic expression for $1-u(x,y;\varepsilon)$.

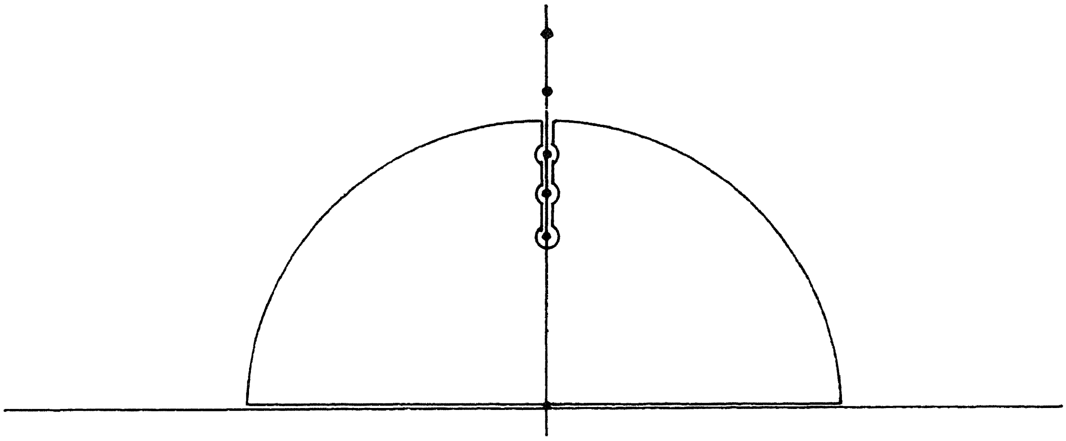
$1-u(r,\theta;1/2M) =$

$$(4.2)\qquad = e^{Mr\cos\theta} \sum_{n=-\infty}^{+\infty} \left[I_n(Mr) - \frac{I_n(M)\, K_n(Mr)}{K_n(M)}\right] e^{in(\pi-\theta)} =$$

$$= e^{Mr\cos\theta} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{I_x(Mr)\, K_x(M) - I_x(M)\, K_x(Mr)}{K_x(M)}\, e^{ix((\pi-\theta)+2\pi m)} dx.$$

The integrand is a single-valued function of x, regarded as a complex variable. The integral can be computed by means of Watson's transformation (cf. WAECHTER [1968]). The singular points of the integrand are found only at the zeros of $K_x(M)$. From section 3.1 we know that, for real M, these zeros of $K_x(M)$ are found only for pure imaginary values of x; $x=\pm i\mu_1, \pm i\mu_2, \dots$.

In the remaining part of this chapter we take $0 \leq \theta \leq \pi$. For $(\pi-\theta) + 2\pi n > 0$ the path of integration is chosen in the upper halfplane $\mathrm{Im}(x) > 0$.



It follows that

$$\int_{-\infty}^{+\infty} \frac{I_x(Mr)\, K_x(M) - I_x(M)\, K_x(Mr)}{K_x(M)}\, e^{ix((\pi-\theta)+2\pi m)} dx = -2\pi i \sum_{j=1,2,3,\ldots} \mathrm{Res}_j$$

with

$$\mathrm{Res}_j = \lim_{x\to i\mu_j} (x-i\mu_j) \frac{I_x(Mr)\, K_x(M) - I_x(M)\, K_x(Mr)}{K_x(M)}\, e^{ix|\pi-\theta+2\pi m|} =$$

$$= - \frac{I_{i\mu_j}(M)\, K_{i\mu_j}(Mr)}{\left[\frac{\partial}{\partial x} K_x(M)\right]_{x=i\mu_j}}\, e^{-\mu_j|\pi-\theta+2\pi m|} .$$

Hence,

$$1-u(r,\theta;1/2M) = -e^{Mr\cos\theta} \sum_{m=-\infty}^{+\infty} -2\pi i \sum_{j=1,2,\ldots} -i \cdot \frac{I_{i\mu_j}(M)\, K_{i\mu_j}(Mr)}{\left[\frac{\partial}{\partial \mu} K_{i\mu}(M)\right]_{\mu=\mu_j}}\, e^{-\mu_j|\pi-\theta+2\pi m|}$$

or

$$u(r,\theta;1/2M) = 1-e^{Mr\cos\theta}\, 2\pi \sum_{j=1,2,\ldots} \frac{I_{i\mu_j}(M)\, K_{i\mu_j}(Mr)}{\left[\frac{\partial}{\partial \mu} K_{i\mu}(M)\right]_{\mu=\mu_j}} \sum_{m=-\infty}^{+\infty} e^{-\mu_j|\pi-\theta+2\pi m|}$$

(4.3)

$$= 1-2\pi\, e^{Mr\cos\theta} \sum_{j=1,2,\ldots} \frac{I_{i\mu_j}(M)\, K_{i\mu_j}(Mr)}{\left[\frac{\partial}{\partial \mu} K_{i\mu}(M)\right]_{\mu=\mu_j}} \frac{\cosh(\mu_j\theta)}{\sinh(\mu_j\pi)} .$$

The expression of the Wronskian

$$I_\nu(z)\, K'_\nu(z) - I'_\nu(z)\, K_\nu(z) = -\frac{1}{z}$$

in particular yields

$$I_{i\mu_j}(M) = -1/(MK'_{i\mu_j}(M)).$$

Hence we obtain the analytic solution of the boundary value problem (1.1), (1.2) in terms of the modified Besselfunction K with pure imaginary argument

$$(4.4)\quad u(r,\theta;1/2M) = 1 + \frac{2\pi}{M} e^{Mr\cos\theta} \sum_{j=1,2,\ldots} \frac{K_{i\mu_j}(Mr)\cosh(\mu_j\theta)}{\left[\frac{\partial}{\partial\mu} K_{i\mu}(M)\right]_{\mu=\mu_j} K'_{i\mu_j}(M)\sinh(\mu_j\pi)}\ .$$

4.2. <u>An asymptotic expansion for the solution</u>

Now we can use the asymptotic expansions mentioned in section 3 in order to find an asymptotic expansion for $u(r,\theta;1/2M)$ for large values of M. Substituting the relevant parts of the sections 3.3, 3.4 and 3.8 for $K_{i\mu}(Mr)$, $\left[\frac{\partial}{\partial\mu} K_{i\mu}(M)\right]_{\mu=\mu_j}$ and $K'_{i\mu}(M)$ repsectively, we obtain from equation (4.4)

$$(4.5)\quad \begin{aligned} u(r,\theta;1/2M) = 1 - 2^{1/6} M^{1/3} e^{Mr\cos\theta}\ \cdot \\ \cdot \sum_{j=1,2,\ldots} \frac{\zeta_j^{1/4}\, Ai(\zeta_j) e^{\pi\mu_j/2}}{(M^2r^2-\mu_j^2)^{1/4}(Ai'(-\alpha_j))^2} \frac{\cosh(\mu_j\theta)}{\sinh(\mu_j\pi)}\ , \end{aligned}$$

where

$$\text{if } Mr \geq \mu_j,\ \text{then } \frac{2}{3}\zeta_j^{3/2} = (M^2r^2-\mu_j^2)^{1/2} - \mu_j \arccos(\mu_j/Mr)\ ;$$

$$\text{if } Mr \leq \mu_j,\ \text{then } \frac{2}{3}(-\zeta_j)^{3/2} = \mu_j \operatorname{arccosh}(\mu_j/Mr) - (\mu_j^2-M^2r^2)^{1/2};$$

- α_j the j-th zero of the Airyfunction Ai(z), and

$$\mu_j = M + \alpha_j \left(\frac{M}{2}\right)^{1/3} + O(M^{-1/3}).$$

Since this formula can be used for the computation of $u(x,y;\varepsilon)$ for values of x,y and ε where (1.17) fails, it is supplementary for numerical purposes.

For the analysis of the behavior of the form (4.5) and in order to obtain more convenient asymptotic expansions we have to distinguish

a. $r-1 \gg O(M^{-2/3})$, and

b. $r-1 = RM^{-2/3}$ with $R = O(1)$.

4.3. An asymptotic expansion for $|\theta| < \pi/2$, $r-1 \gg O(M^{-2/3})$.

First we consider $r-1 \gg O(M^{-2/3})$. In this case $Mr > M + O(M^{1/3})$ and so $Mr > \mu_j$. This enables us to make use of the more specific asymptotic expansion of $K_{i\mu_j}(Mr)$ given in section 3.7 instead of the asymptotic expansion given in section 3.3. Hence we obtain

$$(4.6)\quad u(r,\theta;1/2M) = 1 - \frac{e^{M(r\cos\theta-(r^2-1)^{1/2})}}{2^{5/6}M^{1/6}\pi^{1/2}(r^2-1)^{1/4}} \sum_{j=1,2,..} \frac{e^{\mu_j(\theta-\frac{\pi}{2}+\arccos(\frac{1}{r}))}}{(Ai'(-\alpha_j))^2}.$$

This series is convergent only for

$$\theta - \frac{\pi}{2} + \arccos\left(\frac{1}{r}\right) < 0 \iff \frac{\pi}{2} - \theta > \arccos\left(\frac{1}{r}\right) \iff$$

$$\iff \sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right) < \frac{1}{r}$$

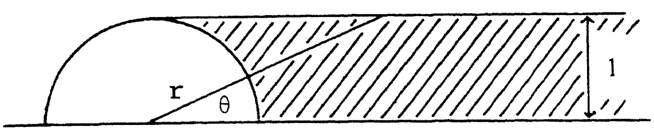


Figure 3. The convergence region for equation (4.6)

Thus this asymptotic expansion describes the behavior of the solution $u(x,y;\varepsilon)$ in the region

$$\{(x,y)\,|\,x^2 + y^2 > 1 + \varepsilon_0,\ x > 0,\ |y| < 1\}$$

with $\varepsilon_0 > 0$, $\varepsilon_0 > \mathcal{O}(M^{-3/2})$, i.e. the region where

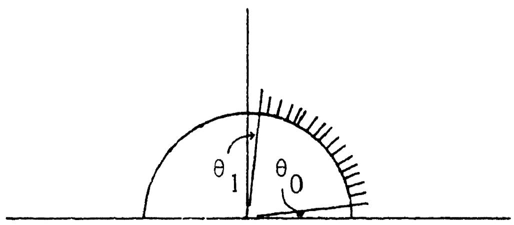
$$\lim_{\varepsilon \downarrow 0} u(x,y;\varepsilon) = 1.$$

4.4. An asymptotic expansion for $|\theta| < \pi/2$, $r-1 = \mathcal{O}(M^{-2/3})$

Next we consider the region near the boundary: $r-1 = RM^{-2/3}$, $R = \mathcal{O}(1)$. In this case $Mr = M+RM^{1/3}$; now we can use (3.11) for $K_{i\mu}(Mr)$. Substituting this asymptotic expansion we obtain

$$(4.7)\qquad u(r,\theta;1/2M) = 1 - e^{Mr\cos\theta} \sum_{j=1,2,\ldots} \frac{Ai(2^{1/3}R-\alpha_j)e^{\mu_j(\theta-\pi/2)}}{\{Ai'(-\alpha_j)\}^2} .$$

This expansion holds for $0 < \theta_0 < \theta < \theta_1 < \frac{\pi}{2}$.



4.5. A Taylor series expansion

When we confine ourselves to the region $r-1 = \mathcal{O}(M^{-1})$, $x > 0$, we can substitute a Taylor series expansion for $K_{i\mu}(Mr)$ in (4.4). This equation then becomes

$$u(r,\theta;1/2M) = 1 + \frac{2\pi}{M} e^{Mr\cos\theta} \sum_{j=1,2,\ldots} \frac{K_{i\mu_j}(M) + M(r-1)K'_{i\mu}(M) + \ldots\}\cosh(\mu_j\theta)}{\left[\frac{\partial}{\partial\mu} K_{i\mu}(M)\right]_{\mu=\mu_j} K'_{i\mu_j}(M)\sinh(\mu_j-\pi)} \approx$$

$$\approx 1 + 2\pi(r-1)e^{Mr\cos\theta} \sum_{j=1,2,\ldots} \frac{\cosh(\mu_j\theta)}{\left[\frac{\partial}{\partial\mu} K_{i\mu}(M)\right]_{\mu=\mu_j} \sinh(\mu_j\pi)}$$

With the asymptotic expansion (3.17) this yields

$$(4.8)\qquad u(r,\theta;1/2M) = 1 - 2^{1/3}M^{2/3}(r-1)e^{Mr\cos\theta}\sum_{j=1,2,\ldots}\frac{\cosh(\mu_j\theta)e^{\pi\mu_j/2}}{Ai'(-\alpha_j)\sinh(\mu_j\pi)} .$$

For $\theta > \theta_0 > 0$ this is approximated by

$$(4.9)\qquad u(r,\theta;1/2M) = 1 - 2^{1/3}M^{2/3}(r-1)e^{Mr\cos\theta}\sum_{j=1,2,\ldots}\frac{e^{\mu_j(\theta-\pi/2)}}{Ai'(-\alpha_j)} .$$

This series is convergent only for $\theta < \pi/2$. The expansion holds for $M \to \infty$

$$r-1 \ll \frac{1}{M} .$$

Of course this result is consistent with and even implied by the result of section 4.4.

4.6. An asymptotic expansion for $x < 0$ or $|y| > 1$

In order to obtain an asymptotic expansion that is valid in the major part of the domain not covered by the expansions given in sections 4.2 to 4.5, we use an integral representation of the solution $u(r,\theta;\varepsilon)$. This was anticipated in the equations (1.7) to (1.9). We consider the following integral, which satisfies the differential equation (1.4),

$$(4.10)\qquad w(r,\theta;M) \equiv \int_C \frac{I_\nu(M)\,K_\nu(Mr)}{K_\nu(M)}\,e^{i\nu\theta}\,d\nu .$$

For the modified Besselfunctions the asymptotic expansions given in (3.12) and (3.13) are substituted. In these expansions we take $\nu z = M$ for $I_\nu(M)$ and $K_\nu(M)$ and we take $\nu z = Mr$ for $K_\nu(Mr)$. This yields, with μ defined by $\nu = M\mu$

$$w(r,\theta;M) \sim \int_G \frac{M}{(\mu^2+r^2)^{1/2}\sqrt{2\pi M}}\exp[M\{i\theta\mu + 2\sqrt{\mu^2+1} - \sqrt{\mu^2+r^2} - 2\mu\,\mathrm{arcsinh}\,\mu +$$

$$+ \mu\,\mathrm{arcsinh}(\tfrac{\mu}{r})\}]d\mu \equiv$$

$$\equiv \int_c f(\mu)\, \exp(Mg(\mu))d\mu$$

for $M \to \infty$.

This integral is approximated by means of the saddlepoint method. The saddlepoint is determined by

$$i\theta - 2\,\mathrm{arcsh}(\mu) + \mathrm{arcsh}(\mu/r) = 0.$$

With α and β defined by

$$-\theta + \beta = 2\alpha$$
$$r \sin \beta = \sin \alpha$$

the saddlepoint is found to be $\mu = -i\sin \alpha$. The path of steepest descent of $g(\mu)$ at the point $\mu = i \sin \alpha$ is given by the line

$$\mathrm{Im}\, \mu = -\sin \alpha.$$

The Taylor series expansion of $g(\mu)$ along this line at the saddlepoint becomes

$$g(\mu) = g(-i\sin\alpha) + \tfrac{1}{2}(\mu + i\sin\alpha)^2\, g''(-i\sin\alpha) + \dots =$$

$$= 2\cos\alpha + r\cos\beta + \tfrac{1}{2}(\mu + i\sin\alpha)^2(-2(\cos\alpha)^{-1} + (r\cos\beta)^{-1}) + \dots .$$

Now

$$f(-i\sin\alpha) = \sqrt{\frac{M}{2\pi r\cos\beta}}$$

and

$$\int_c f(\mu)\, \exp(Mg(\mu))d\mu \sim \qquad (4.11)$$

$$\sim \int_{-\infty-i\sin\alpha}^{+\infty-i\sin\alpha} f(i\sin\alpha)\ \exp\{M[2\cos\alpha+r\cos\beta]-\tfrac{1}{2}M(\mu+i\sin\alpha)^2((r\cos\beta)^{-1}-2(\cos\alpha)^{-1}))d\mu =$$

$$= \sqrt{\frac{\cos\alpha}{2r\cos\beta-\cos\alpha}}\ \exp\{M(2\cos\alpha-r\cos\beta)\}. \tag{4.11}$$

The integrand of (4.10) has its singularities at the zeros of $K_\nu(M)$. As was mentioned in section 3.4 for $M \to \infty$, these zeros can be found at

$$\nu = \pm\ i(M+\alpha_j(\tfrac{M}{2})^{1/3}+O(M^{-1/3})),$$

where $-\alpha_j$ denotes the j-th zero of the Airyfunction. Since $\nu = \mu M$, the singularities are situated in the μ-plane at

$$\mu \sim \pm\ i(1+\alpha_j 2^{-1/3}M^{-2/3}) \qquad \text{for } M \to \infty.$$

Since the saddlepoint is situated at $\mu = -i\sin\alpha$, the singularity and the saddlepoint are separated when $\sin\alpha < 1-\varepsilon$ for some $\varepsilon > 0$.

Hence the asymptotic expansion holds for $|\alpha| < \pi/2 - \delta$ for some $\delta > 0$. Since $\theta \to \alpha$ and $\beta \to \alpha$ for $r \to 1$, also

$$\sqrt{\frac{\cos\alpha}{2r\cos\beta-\cos\alpha}}\ \exp[M(2\cos\alpha+r\cos\beta)] \to \exp[M\cos\theta] \qquad \text{for } r \to 1.$$

This implies that the asymptotic expansion of $w(r,\pi-\theta;M)$ satisfies the boundary conditions of the differential equation (1.4b). Since $w(r,\pi-\theta;M)$ also satisfies the differential equation (1.4a), equation (4.11) yields an asymptotic expansion of $v(r,\pi-\theta;1/2M)$. Hence we obtained an expansion of $u(r,\theta;1/2M)$ in the region $|\alpha| < \pi/2 - \delta$

$$(4.12)\quad u(r,\theta;1/2M) \sim \sqrt{\frac{\cos\alpha}{2r\cos\beta-\cos\alpha}}\ \exp(-M(r\cos\beta-r\cos\theta-2\cos\alpha)),$$

α and β being defined by

$$(4.13)\qquad \begin{aligned} \theta + \beta &= 2\alpha + \pi \\ r\sin\beta &= \sin\alpha. \end{aligned}$$

A geometric interpretation of α and β is given in figure 4.

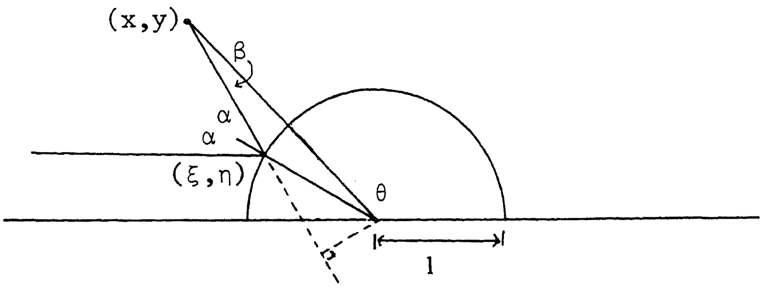


Figure 4

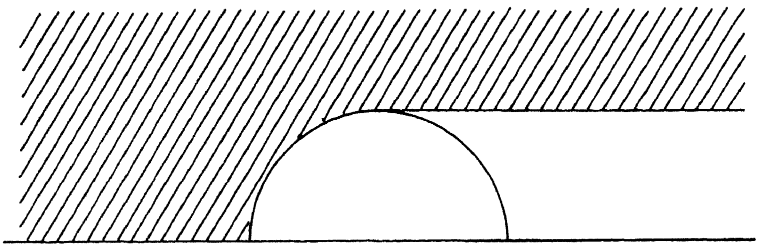


Figure 5. The region of validity of the asymptotic expansion (4.12)

5. BOUNDARY LAYERS

It is not our intention to give a detailed analysis of the boundary-layer structure of the singular perturbation problem under consideration. The interested reader is refered to papers by LUDWIG [1969] and WAECHTER [1968]. In a forthcoming paper we want to obtain a numerical approximation to the boundary value problem (1.1)-(1.2) by some discretization method and for that purpose we need information for the location of an appropriate non-uniform net.

From the analysis given in section 4 it follows that a boundary-layer appears for $x < 0$, $x^2 + y^2 = 1$ and a free boundary-layer appears for

$x > 0$, $y = \pm 1$. From the analysis of the last section the structure of these layers is easily determined.

In the papers mentioned above a more detailed analysis is given for the regions of the birth of the boundary-layers i.e. the neighbourhoods of (0,1) and (0,-1).

5.1. The boundary-layer for $x < 0$, $x^2 + y^2 \downarrow 1$

The exponent of the exponential factor of the asymptotic expansion (4.12) reads

$$(5.1) \qquad -M(-2\cos\alpha + r\cos\beta - r\cos\theta).$$

Hence, an impression of the structure of the boundary-layer is given by the equation (cf.fig.4)

$$(r\cos\beta - \cos\alpha) + (-\cos\alpha - r\cos\theta) = d \qquad \text{or}$$

$$(5.2) \qquad \sqrt{(x-\xi)^2+(y-\eta)^2} + (\xi - x) = d.$$

Since $\xi - x = \cos 2\alpha \sqrt{(x-\xi)^2 + (y-\eta)^2}$, it follows that the thickness of the boundary-layer in the neighborhood of $x^2 + y^2 = 1$, $x < \varepsilon_0 < 0$, is given by

$$(5.3) \qquad \sqrt{(x-\xi)^2 + (y-\eta)^2}\cos\alpha = \frac{d\cos\alpha}{1+\cos 2\alpha} = \frac{d}{2\cos\alpha} \approx \frac{d}{2}\frac{1}{|\cos\theta|}.$$

5.2. The boundary-layer for $x > 0$, $y \downarrow 1$

The structure of this boundary-layer is also determined from (5.2)

$$|x-\xi| \sqrt{1 + (\tan 2\alpha)^2} = d + (x-\xi)$$

$$x > 0 \Rightarrow \alpha > \pi/4 \Rightarrow x - \xi > 0 \Rightarrow (x-\xi) = \frac{-d\cos 2\alpha}{1+\cos 2\alpha} = \frac{d}{2}(1-\tan^2\alpha)$$

$$(5.4) \qquad y - \eta = (x-\xi)\tan(2\alpha) = d\tan(\alpha) \Rightarrow d^2 + 2d(x-\xi) = (y-\eta)^2.$$

Thus, for this boundary-layer ($\alpha \sim \pi/2 \to \xi \sim 0$ $\eta \sim 1$) the structure is given by

$$(5.5) \qquad 2dx \sim (y-1)^2$$

(Notice the resemblance between the equations (2.4) and (5.4)).

5.3. The boundary-layer for $x > 0$, $y \uparrow 1$

The structure of this boundary-layer is calculated from the exponential factor in the asymptotic expansion (4.6)

$$(5.6) \qquad r\cos\theta - \sqrt{r^2 - 1} + \theta - \frac{\pi}{2} + \arccos\left(\frac{1}{r}\right) = -d.$$

This also yields

$$(5.7) \qquad 2dx \approx (1-y)^2$$

when $x > \varepsilon_1 > 0$, $0 < 1 - y << 1$.

5.4. The behavior in the boundary-layer $x > \varepsilon_1 > 0$, $y \approx 1$

In chapter 4 we found that for $x > \varepsilon_1 > 0$

$$\lim_{\varepsilon \downarrow 0} u(x,y;\varepsilon) = \begin{cases} 0 & \text{for } 1 + \varepsilon_2 < y \\ 1 & \text{for } 0 \le y < 1 - \varepsilon_2 \end{cases}$$

for some $\varepsilon_2 > 0$. No description was given of the behavior in the boundary-layer region.

Using a standard argument, we assume that, for $x > \varepsilon_1 > 0$, $y \approx 1$, the values of $\partial^2 u/\partial y^2$ are much larger than the values of $\partial^2 u/\partial x^2$. Hence, the equation (1.1) is approximated by

$$\varepsilon\, \partial^2 u/\partial y^2 = \partial u/\partial x.$$

The solution of this differential equation which matches

$$\lim_{\varepsilon \downarrow 0} u(x,y;\varepsilon)$$

in both regions $y > 1$ and $y < 1$, is given by

$$u(x,y;\ ) = \tfrac{1}{2} + \tfrac{1}{2}\,\mathrm{erf}\left(\frac{1-y}{2\sqrt{x\varepsilon}}\right) = \tag{5.8}$$

$$= \tfrac{1}{2}\,\mathrm{erf\,c}\left(\frac{y-1}{2\sqrt{x\varepsilon}}\right)$$

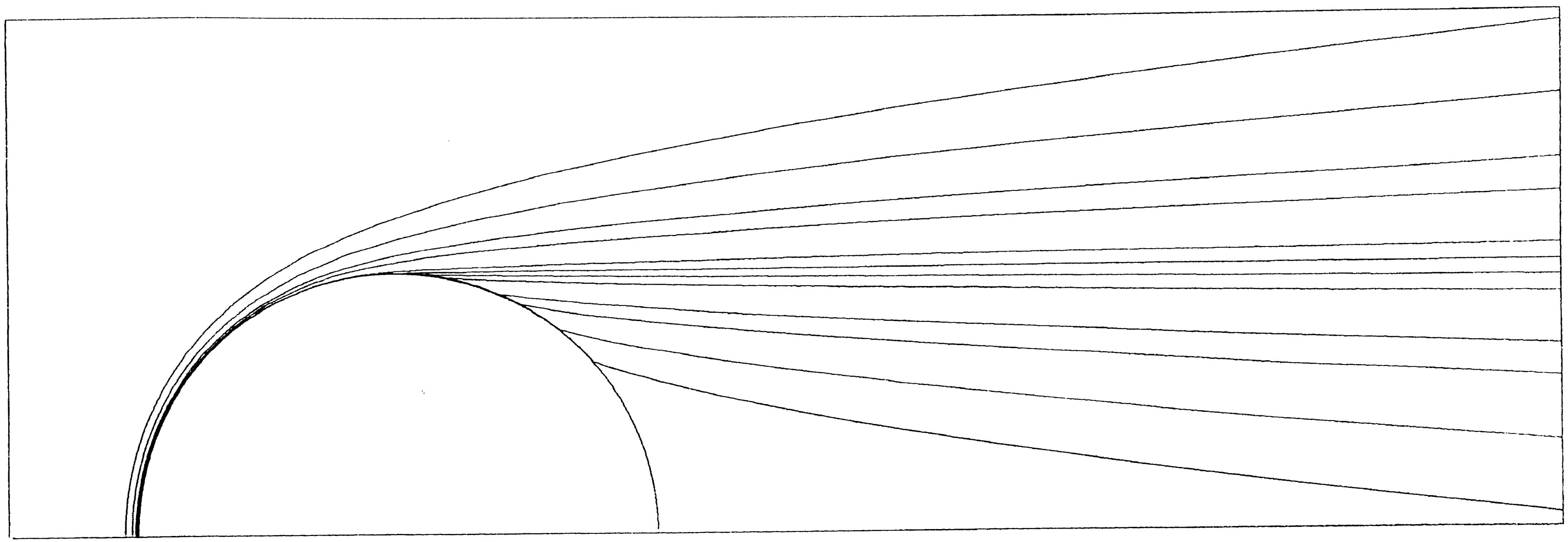


Figure 6. The boundary layer behavior of $u(x,y;\varepsilon)$. Lines with constant exponential factor d (c.f. equations (5.2) and (5.6)).

6. NUMERICAL RESULTS

In section 2 we mentioned that the evaluation of the expression (1.12) failed for small ε, particularly in the neighborhood of the point (1,0). With the help of the asymptotic expansions given in the foregoing sections, it is now possible to evaluate the function $u(x,y;\varepsilon)$, for small values of ε, in a larger domain. Theoretically we have to our disposal asymptotic expressions for $\varepsilon \to 0$, that are valid on $x^2 + y^2 \geq 1$ with exception of two small regions with diameter $O(\varepsilon^{1/3})$ at (0,1) and 0,-1), i.e. the regions of the birth of the boundary-layer. However, due to cancellation of digits, also the asymptotic expressions are not suitable for numerical evaluation everywhere in the domain where they have been shown to converge.

By a number of computational results, we will compare the numerical properties of the four different expressions for $u(x,y;\varepsilon)$ already obtained.

We denote the numerical values of $u(x,y;\varepsilon)$ obtained by evaluation of expression (1.12) by $u1(x,y;\varepsilon)$, the values obtained by evaluations of expression (4.5) by $u2(x,y;\varepsilon)$ and the values obtained by (4.12) by $u3(x,y;\varepsilon)$. In addition, expression (5.9) is used for the following approximation of the behavior in the boundary-layer

$$u4(x,y;\varepsilon) = u2(x,1;\varepsilon)\ \mathrm{erfc}\left(\frac{y-1}{2\sqrt{x\varepsilon}}\right).$$

Table 3. ul(x,y;ε); ε = 0.1; relative error < $4.5\cdot 10^{-9}$

For ε > 0.05 the analytic expression (1.12) delivers sufficiently accurate numerical results.

y \ x	-2.00	-1.00	+0.00	+1.00	+2.00	+3.00	+4.00	+5.00	+6.00
+2.00	+1.34504" -7	+1.63522" -4	+1.29065" -2	+7.48931" -2	+1.42183" -1	+1.90612" -1	+2.24830" -1	+2.49908" -1	+2.68833" -1
+1.90	+2.04560" -7	+2.74936" -4	+2.09002" -2	+1.04156" -1	+1.78837" -1	+2.27247" -1	+2.59711" -1	+2.82740" -1	+2.99656" -1
+1.80	+3.08651" -7	+4.62426" -4	+3.36951" -2	+1.42535" -1	+2.21718" -1	+2.67919" -1	+2.97376" -1	+3.17586" -1	+3.31971" -1
+1.70	+4.61616" -7	+7.77975" -4	+5.40236" -2	+1.91593" -1	+2.70798" -1	+3.12338" -1	+3.37527" -1	+3.54174" -1	+3.65537" -1
+1.60	+6.83625" -7	+1.30901" -3	+8.60153" -2	+2.52475" -1	+3.25687" -1	+3.60035" -1	+3.79768" -1	+3.92173" -1	+4.00071" -1
+1.50	+1.00135" -6	+2.20231" -3	+1.35737" -1	+3.25506" -1	+3.85581" -1	+4.10368" -1	+4.23612" -1	+4.31197" -1	+4.35250" -1
+1.40	+1.44891" -6	+3.70372" -3	+2.11733" -1	+4.09736" -1	+4.49259" -1	+4.62536" -1	+4.68496" -1	+4.70812" -1	+4.70719" -1
+1.30	+2.06813" -6	+6.22326" -3	+3.25232" -1	+5.02568" -1	+5.15136" -1	+5.15622" -1	+5.13796" -1	+5.10545" -1	+5.06100" -1
+1.20	+2.90758" -6	+1.04404" -2	+4.89217" -1	+5.99641" -1	+5.81364" -1	+5.68626" -1	+5.58850" -1	+5.49901" -1	+5.40993" -1
+1.10	+4.01961" -6	+1.74697" -2	+7.14577" -1	+6.95155" -1	+6.45991" -1	+6.20532" -1	+6.02983" -1	+5.88370" -1	+5.74990" -1
+1.00	+5.45457" -6	+2.91117" -2	+1.00000" +0	+7.82747" -1	+7.07134" -1	+6.70353" -1	+6.45529" -1	+6.25444" -1	+6.07681" -1
+0.90	+7.25169" -6	+4.82042" -2		+8.56811" -1	+7.63169" -1	+7.17190" -1	+6.85851" -1	+6.60628" -1	+6.38664" -1
+0.80	+9.42671" -6	+7.90500" -2		+9.13865" -1	+8.12871" -1	+7.60272" -1	+7.23367" -1	+6.93451" -1	+6.67551" -1
+0.70	+1.19576" -5	+1.27769" -1		+9.53386" -1	+8.55505" -1	+7.98984" -1	+7.57558" -1	+7.23476" -1	+6.93976" -1
+0.60	+1.47709" -5	+2.02147" -1		+9.77660" -1	+8.90835" -1	+8.32880" -1	+7.87978" -1	+7.50310" -1	+7.17603" -1
+0.50	+1.77336" -5	+3.10083" -1		+9.90689" -1	+9.19065" -1	+8.61673" -1	+8.14260" -1	+7.73606" -1	+7.38129" -1
+0.40	+2.06542" -5	+4.55408" -1		+9.96703" -1	+9.40720" -1	+8.85215" -1	+8.36113" -1	+7.93067" -1	+7.55293" -1
+0.30	+2.32984" -5	+6.30581" -1		+9.99045" -1	+9.56502" -1	+9.03463" -1	+8.53315" -1	+8.08453" -1	+7.68875" -1
+0.20	+2.54187" -5	+8.09312" -1		+9.99790" -1	+9.67136" -1	+9.16441" -1	+8.65706" -1	+8.19577" -1	+7.78701" -1
+0.10	+2.67940" -5	+9.47461" -1		+9.99972" -1	+9.73240" -1	+9.24197" -1	+8.73181" -1	+8.26304" -1	+7.84648" -1
+0.00	+2.72708" -5	+1.00000" +0		+1.00000" +0	+9.75227" -1	+9.26777" -1	+8.75679" -1	+8.28555" -1	+7.86638" -1

Table 4. $u2(x,y;\varepsilon)$; $\varepsilon = 0.1$.

The asymptotic expansion (4.5) converges also for some $y > 1$.

y \ x	-2.00	-1.00	+0.00	+1.00	+2.00	+3.00	+4.00	+5.00	+6.00
+2.00	+1.00000" +0	+9.99987" -1	+7.95521" -1	+7.10273" -1	+1.79078" -1	+1.96161" -1	+2.24330" -1	+2.48922" -1	+2.67826" -1
+1.90	+1.00000" +0	+9.99984" -1	+3.98086" -1	+6.58417" -1	+2.21076" -1	+2.31336" -1	+2.59109" -1	+2.81693" -1	+2.98602" -1
+1.80	+1.00000" +0	+9.99969" -1	+3.89101" -1	+3.08235" -1	+2.60152" -1	+2.70544" -1	+2.96611" -1	+3.16475" -1	+3.30871" -1
+1.70	+1.00000" +0	+9.99972" -1	+3.35097" -1	+3.83326" -1	+3.01436" -1	+3.13738" -1	+3.36589" -1	+3.53001" -1	+3.64392" -1
+1.60	+1.00000" +0	+9.99974" -1	+3.41652" -1	+4.32422" -1	+3.47757" -1	+3.61543" -1	+3.78670" -1	+3.90940" -1	+3.98882" -1
+1.50	+1.00000" +0	+9.99976" -1	+5.11168" -1	+4.71041" -1	+4.00034" -1	+4.11393" -1	+4.22378" -1	+4.29910" -1	+4.34021" -1
+1.40	+1.00000" +0	+9.99978" -1	+6.56485" -1	+5.12083" -1	+4.68574" -1	+4.62909" -1	+4.67155" -1	+4.69477" -1	+4.69454" -1
+1.30	+1.00000" +0	+9.99980" -1	+7.72550" -1	+5.63670" -1	+5.33221" -1	+5.15375" -1	+5.12397" -1	+5.09171" -1	+5.04803" -1
+1.20	+1.00000" +0	+9.99988" -1	+8.58932" -1	+7.31951" -1	+5.95546" -1	+5.67878" -1	+5.57395" -1	+5.48495" -1	+5.39668" -1
+1.10	+1.00000" +0	+9.99993" -1	+9.18676" -1	+8.48574" -1	+6.55794" -1	+6.19420" -1	+6.01486" -1	+5.86942" -1	+5.73643" -1
+1.00	+1.00000" +0	+9.99996" -1	+9.56868" -1	+9.19416" -1	+7.13146" -1	+6.69004" -1	+6.44005" -1	+6.24001" -1	+6.06316" -1
+0.90	+1.00000" +0	+9.99998" -1		+9.59578" -1	+7.66338" -1	+7.15844" -1	+6.84321" -1	+6.59178" -1	+6.37287" -1
+0.80	+1.00000" +0	+9.99999" -1		+9.80856" -1	+8.14131" -1	+7.58917" -1	+7.21846" -1	+6.92002" -1	+6.66165" -1
+0.70	+1.00000" +0	+1.00000" +0		+9.91412" -1	+8.55611" -1	+7.97621" -1	+7.56059" -1	+7.22033" -1	+6.92585" -1
+0.60	+1.00000" +0	+1.00000" +0		+9.96330" -1	+8.91514" -1	+8.31531" -1	+7.86510" -1	+7.48878" -1	+7.16210" -1
+0.50	+1.00000" +0	+1.00000" +0		+9.98493" -1	+9.19923" -1	+8.60362" -1	+8.12830" -1	+7.72186" -1	+7.36737" -1
+0.40	+1.00000" +0	+1.00000" +0		+9.99398" -1	+9.41339" -1	+8.83956" -1	+8.34720" -1	+7.91662" -1	+7.53903" -1
+0.30	+1.00000" +0	+1.00000" +0		+9.99761" -1	+9.56817" -1	+9.02260" -1	+8.51956" -1	+8.07061" -1	+7.67487" -1
+0.20	+1.00000" +0	+1.00000" +0		+9.99902" -1	+9.67215" -1	+9.15286" -1	+8.64374" -1	+8.18195" -1	+7.77316" -1
+0.10	+1.00000" +0	+1.00000" +0		+9.99954" -1	+9.73181" -1	+9.23076" -1	+8.71866" -1	+8.24928" -1	+7.83264" -1
+0.00	+1.00000" +0	+1.00000" +0		+9.99967" -1	+9.75123" -1	+9.25667" -1	+8.74370" -1	+8.27182" -1	+7.85255" -1

no convergence

convergence

Table 5. $u1(x,y;\varepsilon)$; $\varepsilon = 0.05$; relative error $< 10^{-4}$.

The analytic expression (1.12) becomes soon worse for $\leq$ 0.05.

y \ x	-2.00	-1.00	+0.00	+1.00	+2.00	+3.00	+4.00	+5.00	+6.00
+2.00	+3.77100"-14	+4.98688" -8	+3.00726" -4	+1.06204" -2	+4.01275" -2	+7.41173" -2	+1.04758" -1	+1.30864" -1	+1.52947" -1
+1.90	+8.58036"-14	+1.37291" -7	+7.63449" -4	+1.99373" -2	+6.17009" -2	+1.02439" -1	+1.36037" -1	+1.63217" -1	+1.85450" -1
+1.80	+1.92146"-13	+3.77764" -7	+1.91604" -3	+3.61592" -2	+9.20059" -2	+1.38262" -1	+1.73386" -1	+2.00497" -1	+2.22014" -1
+1.70	+4.22733"-13	+1.03861" -6	+4.74104" -3	+6.31125" -2	+1.32894" -1	+1.82195" -1	+2.16912" -1	+2.42612" -1	+2.62464" -1
+1.60	+9.11908"-13	+2.85216" -6	+1.15250" -2	+1.05568" -1	+1.85753" -1	+2.34387" -1	+2.66400" -1	+2.89242" -1	+3.06462" -1
+1.50	+1.92456"-12	+7.81928" -6	+2.73902" -2	+1.68482" -1	+2.51064" -1	+2.94397" -1	+3.21270" -1	+3.39833" -1	+3.53507" -1
+1.40	+3.96412"-12	+2.13859" -5	+6.32131" -2	+2.55418" -1	+3.28001" -1	+3.61117" -1	+3.80573" -1	+3.93604" -1	+4.02945" -1
+1.30	+7.94749"-12	+5.82946" -5	+1.40300" -1	+3.66299" -1	+4.14205" -1	+4.32795" -1	+4.43031" -1	+4.49579" -1	+4.53995" -1
+1.20	+1.54633"-11	+1.58148" -4	+2.95154" -1	+4.95308" -1	+5.05862" -1	+5.07146" -1	+5.07113" -1	+5.06633" -1	+5.05774" -1
+1.10	+2.91056"-11	+4.26161" -4	+5.75294" -1	+6.30416" -1	+5.98150" -1	+5.81569" -1	+5.71137" -1	+5.63557" -1	+5.57343" -1
+1.00	+5.28151"-11	+1.13739" -3	+1.00000" +0	+7.55886" -1	+6.85985" -1	+6.53417" -1	+6.33411" -1	+6.19127" -1	+6.07748" -1
+0.90	+9.20584"-11	+2.99408" -3		+8.57516" -1	+7.64881" -1	+7.20284" -1	+6.92353" -1	+6.72172" -1	+6.56053" -1
+0.80	+1.53547"-10	+7.72670" -3		+9.28052" -1	+8.31689" -1	+7.80253" -1	+7.46610" -1	+7.21646" -1	+7.01386" -1
+0.70	+2.44115"-10	+1.93762" -2		+9.69231" -1	+8.84984" -1	+8.32058" -1	+7.95140" -1	+7.66652" -1	+7.42978" -1
+0.60	+3.68495"-10	+4.66214" -2		+9.89085" -1	+9.25031" -1	+8.75132" -1	+8.37240" -1	+8.06491" -1	+7.80158" -1
+0.50	+5.26145"-10	+1.05743" -1		+9.96853" -1	+9.53378" -1	+9.09553" -1	+8.72553" -1	+8.40660" -1	+8.12383" -1
+0.40	+7.08067"-10	+2.20804" -1		+9.99279" -1	+9.72284" -1	+9.35882" -1	+9.01006" -1	+8.68822" -1	+8.39228" -1
+0.30	+8.95285"-10	+4.12186" -1		+9.99875" -1	+9.84133" -1	+9.54979" -1	+9.22735" -1	+8.90801" -1	+8.60387" -1
+0.20	+1.06076" -9	+6.65666" -1		+9.99982" -1	+9.91027" -1	+9.67773" -1	+9.37981" -1	+9.06505" -1	+8.75633" -1
+0.10	+1.17536" -9	+9.01343" -1		+1.00000" +0	+9.94527" -1	+9.75079" -1	+9.46998" -1	+9.15929" -1	+8.84836" -1
+0.00	+1.21642" -9	+1.00000" +0		+1.00000" +0	+9.95599" -1	+9.77450" -1	+9.49981" -1	+9.19066" -1	+8.87908" -1

Table 6. $u3(x,y;\varepsilon)$ or $u2(x,y;\varepsilon)$; $\varepsilon = 0.05$.

Although not very accurate, the asymptotic expansions (4.5) and (4.12) permit numerical results.

y \ x	-2.00	-1.00	+0.00	+1.00	+2.00	+3.00	+4.00	+5.00	+6.00
+2.00	+3.62784"-14	+4.69302" -8	+2.60137" -4	+7.48114" -3	+2.16420" -2	+8.65746" -1	+8.63603" -1	+8.65655" -1	+1.52578" -1
+1.90	+8.26303"-14	+1.29356" -7	+6.56364" -4	+1.35968" -2	+3.14250" -2	+8.38887" -1	+8.43324" -1	+1.62828" -1	+1.85066" -1
+1.80	+1.85233"-13	+3.56411" -7	+1.63508" -3	+2.37027" -2	+4.38038" -2	+1.38030" -1	+1.72986" -1	+2.00088" -1	+2.21613" -1
+1.70	+4.07956"-13	+9.81372" -7	+4.00887" -3	+3.93838" -2	+1.38953" -1	+1.81912" -1	+2.16483" -1	+2.42179" -1	+2.62044" -1
+1.60	+8.80981"-13	+2.69954" -6	+9.63279" -3	+6.18935" -2	+1.89214" -1	+2.34019" -1	+2.65934" -1	+2.88782" -1	+3.06022" -1
+1.50	+1.86131"-12	+7.41508" -6	+2.25505" -2	+2.65881" -1	+2.54538" -1	+2.93934" -1	+3.20762" -1	+3.39345" -1	+3.53045" -1
+1.40	+3.83806"-12	+2.03247" -5	+5.09940" -2	+3.13366" -1	+3.30285" -1	+3.60577" -1	+3.80022" -1	+3.93088" -1	+4.02463" -1
+1.30	+7.70322"-12	+5.55404" -5	+1.09948" -1	+4.52136" -1	+4.15061" -1	+4.32173" -1	+4.42441" -1	+4.49038" -1	+4.53495" -1
+1.20	+1.50044"-11	+1.51110" -4	+2.21262" -1	+5.69136" -1	+5.06425" -1	+5.06453" -1	+5.06489" -1	+5.06071" -1	+5.05261" -1
+1.10	+2.82721"-11	+4.08545" -4	+3.99375" -1	+6.71584" -1	+5.98311" -1	+5.80827" -1	+5.70492" -1	+5.62982" -1	+5.56821" -1
+1.00	+5.13561"-11	+1.09454" -3	+9.56970" -1	+7.71529" -1	+6.85606" -1	+6.52654" -1	+6.32758" -1	+6.18547" -1	+6.07220" -1
+0.90	+8.96053"-11	+2.89387" -3		+9.07816" -1	+7.64194" -1	+7.19529" -1	+6.91708" -1	+6.71598" -1	+6.55526" -1
+0.80	+1.49597"-10	+7.50488" -3		+9.70579" -1	+8.30964" -1	+7.79534" -1	+7.45988" -1	+7.21083" -1	+7.00869" -1
+0.70	+2.38046"-10	+1.89215" -2		+9.91641" -1	+8.84385" -1	+8.31400" -1	+7.94552" -1	+7.66111" -1	+7.42471" -1
+0.60	+3.59623"-10	+4.57835" -2		+9.97858" -1	+9.24515" -1	+8.74554" -1	+8.36699" -1	+8.05979" -1	+7.79665" -1
+0.50	+5.13841"-10	+1.04411" -1		+9.99497" -1	+9.52967" -1	+9.09060" -1	+8.72064" -1	+8.40177" -1	+8.11908" -1
+0.40	+6.91923"-10	+2.19071" -1		+9.99889" -1	+9.71980" -1	+9.35475" -1	+9.00569" -1	+8.68372" -1	+8.38773" -1
+0.30	+8.75293"-10	+4.10472" -1		+9.99976" -1	+9.83924" -1	+9.54645" -1	+9.22343" -1	+8.90376" -1	+8.59946" -1
+0.20	+1.03743" -9	+6.64527" -1		+9.99995" -1	+9.90885" -1	+9.67497" -1	+9.37624" -1	+9.06106" -1	+8.75208" -1
+0.10	+1.14976" -9	+9.00982" -1		+9.99999" -1	+9.94430" -1	+9.74841" -1	+9.46666" -1	+9.15541" -1	+8.84416" -1
+0.00	+1.19001" -9	+1.00000" +0		+9.99999" -1	+9.95506" -1	+9.77224" -1	+9.49657" -1	+9.18684" -1	+8.87493" -1

$u3(x,y;\varepsilon)$ $u2(x,y;\varepsilon)$

Table 7. u2(x,y;ε); ε = 0.01.

The convergence region for the asymptotic expansion (4.5) decreases, for $\varepsilon \to 0$, to $x > 0$, $|y| < 1$.

y \ x	-2.00	-1.00	+0.00	+1.00	+2.00	+3.00	+4.00	+5.00	+6.00
+2.00	+1.00000" +0	+1.00000" +0	+1.00000" +0	+1.00000" +0	+9.99952" -1	+9.99501" -1	+9.98244" -1	+9.96219" -1	+9.93722" -1
+1.90	+1.00000" +0	+1.00000" +0	+1.00000" +0	+9.99996" -1	+9.99723" -1	+9.98241" -1	+9.95342" -1	+9.91650" -1	+9.87781" -1
+1.80	+1.00000" +0	+1.00000" +0	+1.00000" +0	+9.99954" -1	+9.98668" -1	+9.94612" -1	+9.88967" -1	+9.83207" -1	+9.78040" -1
+1.70	+1.00000" +0	+1.00000" +0	+1.00000" +0	+9.99582" -1	+9.94700" -1	+9.85715" -1	+9.76708" -1	+9.69285" -1	+9.63579" -1
+1.60	+1.00000" +0	+1.00000" +0	+9.99998" -1	+9.97028" -1	+9.82690" -1	+9.67369" -1	+9.56266" -1	+9.48957" -1	+9.44292" -1
+1.50	+1.00000" +0	+1.00000" +0	+9.99934" -1	+9.84164" -1	+9.54081" -1	+9.36022" -1	+9.27092" -1	+9.23009" -1	+8.43172" -1
+1.40	+1.00000" +0	+1.00000" +0	+9.98402" -1	+9.38919" -1	+9.02056" -1	+8.92735" -1	+8.92272" -1	+1.48973" -1	+1.71266" -1
+1.30	+1.00000" +0	+1.00000" +0	+9.73499" -1	+8.36260" -1	+8.33660" -1	+1.74102" -1	+2.08921" -1	+2.34773" -1	+2.54867" -1
+1.20	+1.00000" +0	+1.00000" +0	+7.45789" -1	+1.76249" -1	+2.57628" -1	+2.99132" -1	+3.24852" -1	+3.42735" -1	+3.56079" -1
+1.10	+1.00000" +0	+1.00000" +0	+5.57492" -1	+4.14032" -1	+4.43070" -1	+4.54680" -1	+4.61255" -1	+4.65616" -1	+4.68777" -1
+1.00	+1.00000" +0	+1.00000" +0	+9.57046" -1	+7.00292" -1	+6.43890" -1	+6.18221" -1	+6.02719" -1	+5.92062" -1	+5.84157" -1
+0.90	+1.00000" +0	+1.00000" +0		+9.02402" -1	+8.11854" -1	+7.63240" -1	+7.31931" -1	+7.09641" -1	+6.92741" -1
+0.80	+1.00000" +0	+1.00000" +0		+9.81775" -1	+9.19772" -1	+8.71534" -1	+8.35809" -1	+8.08415" -1	+7.86659" -1
+0.70	+1.00000" +0	+1.00000" +0		+9.98215" -1	+9.72839" -1	+9.39571" -1	+9.09286" -1	+8.83361" -1	+8.61310" -1
+0.60	+1.00000" +0	+1.00000" +0		+9.99912" -1	+9.92774" -1	+9.75520" -1	+9.55007" -1	+9.34720" -1	+9.15836" -1
+0.50	+1.00000" +0	+1.00000" +0		+9.99999" -1	+9.98497" -1	+9.91496" -1	+9.80036" -1	+9.66508" -1	+9.52434" -1
+0.40	+1.00000" +0	+1.00000" +0		+1.00000" +0	+9.99756" -1	+9.97474" -1	+9.92095" -1	+9.84282" -1	+9.75005" -1
+0.30	+1.00000" +0	+1.00000" +0		+1.00000" +0	+9.99969" -1	+9.99359" -1	+9.97211" -1	+9.93258" -1	+9.87784" -1
+0.20	+1.00000" +0	+1.00000" +0		+1.00000" +0	+9.99997" -1	+9.99861" -1	+9.99121" -1	+9.97342" -1	+9.94385" -1
+0.10	+1.00000" +0	+1.00000" +0		+1.00000" +0	+1.00000" +0	+9.99974" -1	+9.99742" -1	+9.98971" -1	+9.97372" -1
+0.00	+1.00000" +0	+1.00000" +0		+1.00000" +0	+1.00000" +0	+9.99992" -1	+9.99878" -1	+9.99390" -1	+9.98210" -1

no convergence

convergence

Table 8. u3(x,y;ε) or u4(x,y;ε); ε = 0.01.

The region where the asymptotic expansion (4.12) can be used, increases for ε → 0 to the complement of x > 0, |y| ≤ 1.

y \ x	-2,00	-1,00	+0,00	+1,00	+2,00	+3,00	+4,00	+5,00	+6,00
+2,00	+1,74765"-66	+4,81218"-36	+3,93121"-17	+2,99705" -9	+2,67207" -6	+5,49877" -5	+2,73449" -4	+9,26814" -4	+2,27378" -3
+1,90	+9,93803"-65	+6,80515"-34	+3,64003"-15	+6,12103" -8	+1,94451" -5	+2,27396" -4	+8,81607" -4	+2,62078" -3	+5,47633" -3
+1,80	+5,21556"-63	+9,54421"-32	+3,14606"-13	+1,03385" -6	+1,18419" -4	+8,13639" -4	+2,81936" -3	+6,75663" -3	+1,22213" -2
+1,70	+2,50439"-61	+1,32517"-29	+2,49740"-11	+1,40850" -5	+5,96460" -4	+2,63778" -3	+8,03324" -3	+1,59008" -2	+2,52987" -2
+1,60	+1,08973"-59	+1,81732"-27	+1,78153" -9	+1,50376" -4	+2,45358" -3	+8,84420" -3	+2,04291" -2	+3,42091" -2	+4,86395" -2
+1,50	+4,25108"-58	+2,45427"-25	+1,10799" -7	+1,21644" -3	+8,12874" -3	+2,54873" -2	+4,64696" -2	+6,74040" -2	+8,69895" -2
+1,40	+1,46927"-56	+3,25112"-23	+5,75161" -6	+7,16465" -3	+2,92972" -2	+6,33494" -2	+9,48072" -2	+1,21907" -1	+1,44995" -1
+1,30	+4,44081"-55	+4,20218"-21	+2,33292" -4	+2,92574" -2	+8,60330" -2	+1,36424" -1	+1,74092" -1	+2,02948" -1	+2,25763" -1
+1,20	+1,15713"-53	+5,26189"-19	+6,63959" -3	+1,10155" -1	+2,04313" -1	+2,56077" -1	+2,89004" -1	+3,12069" -1	+3,29291" -1
+1,10	+2,55943"-52	+6,32053"-17	+1,08512" -1	+3,35790" -1	+3,97328" -1	+4,22302" -1	+4,36172" -1	+4,45129" -1	+4,51454" -1
+1,00	+4,72636"-51	+7,18327"-15	+0,00000" +0	+7,00292" -1	+6,43890" -1	+6,18221" -1	+6,02719" -1	+5,92062" -1	+5,84157" -1
+0,90	+7,15960"-50	+7,57451"-13		+1,06479" +0	+8,90451" -1	+8,14141" -1	+7,69266" -1	+7,38994" -1	+7,16860" -1
+0,80	+8,73469"-49	+7,20558"-11		+1,29043" +0	+1,08347" +0	+9,80365" -1	+9,16434" -1	+8,72054" -1	+8,39023" -1
+0,70	+8,42231"-48	+5,93895" -9		+1,37685" +0	+1,20175" +0	+1,10002" +0	+1,03135" +0	+9,81175" -1	+9,42551" -1
+0,60	+6,29897"-47	+4,00199" -7		+1,39731" +0	+1,25848" +0	+1,17309" +0	+1,11063" +0	+1,06222" +0	+1,02332" +0
+0,50	+3,58776"-46	+2,03176" -5		+1,40030" +0	+1,27978" +0	+1,21096" +0	+1,15897" +0	+1,11672" +0	+1,08132" +0
+0,40	+1,52989"-45	+6,95774" -4		+1,40057" +0	+1,28604" +0	+1,22760" +0	+1,18501" +0	+1,14991" +0	+1,11967" +0
+0,30	+4,80993"-45	+1,39752" -2		+1,40058" +0	+1,28748" +0	+1,23380" +0	+1,19740" +0	+1,16822" +0	+1,14301" +0
+0,20	+1,10081"-44	+1,40462" -1		+1,40058" +0	+1,28774" +0	+1,23577" +0	+1,20262" +0	+1,17737" +0	+1,15609" +0
+0,10	+1,81635"-44	+6,05740" -1		+1,40058" +0	+1,28778" +0	+1,23630" +0	+1,20456" +0	+1,18150" +0	+1,16284" +0
+0,00	+2,14779"-44	+1,00000" +0		+1,40058" +0	+1,28778" +0	+1,23642" +0	+1,20519" +0	+1,18320" +0	+1,16604" +0

u3(x,y;ε) u4(x,y;ε)

Table 9. u2(x,y;ε); ε = 0.001.

accuracy: u2(6,1.1,0.001) ca. $2_{10}{-3}$,
u2(6,1.0,0.001) ca. $7.5_{10}{-5}$,
u2(6,0 ,0.001) ca. $_{10}{-14}$.

y \ x	-2,00	-1,00	+0,00	+1,00	+2,00	+3,00	+4,00	+5,00	+6,00
+2,00	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0
+1,90	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0
+1,80	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0
+1,70	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0
+1,60	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0
+1,50	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+9,99998" -1	+9,99990" -1
+1,40	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+9,99998" -1	+9,99977" -1	+9,99890" -1	+9,99689" -1
+1,30	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+9,99979" -1	+9,99687" -1	+9,98798" -1	+9,97348" -1	+9,95566" -1
+1,20	+1,00000" +0	+1,00000" +0	+1,00000" +0	+9,99777" -1	+9,95596" -1	+9,88250" -1	+9,81452" -1	+9,76172" -1	+9,72295" -1
+1,10	+1,00000" +0	+1,00000" +0	+9,99922" -1	+9,34621" -1	+9,13879" -1	+1,37182" -1	+1,72409" -1	+1,99855" -1	+2,22347" -1
+1,00	+1,00000" +0	+1,00000" +0	+9,57062" -1	+6,38998" -1	+5,99135" -1	+5,81190" -1	+5,70424" -1	+5,63063" -1	+5,57662" -1
+0,90	+1,00000" +0	+1,00000" +0		+9,96371" -1	+9,68256" -1	+9,34129" -1	+9,03609" -1	+8,77677" -1	+8,55714" -1
+0,80	+1,00000" +0	+1,00000" +0		+1,00000" +0	+9,99740" -1	+9,97545" -1	+9,92427" -1	+9,84962" -1	+9,76047" -1
+0,70	+1,00000" +0	+1,00000" +0		+1,00000" +0	+1,00000" +0	+9,99981" -1	+9,99812" -1	+9,99252" -1	+9,98107" -1
+0,60	+1,00000" +0	+1,00000" +0		+1,00000" +0	+1,00000" +0	+1,00000" +0	+9,99999" -1	+9,99986" -1	+9,99931" -1
+0,50	+1,00000" +0	+1,00000" +0		+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+9,99999" -1
+0,40	+1,00000" +0	+1,00000" +0		+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0
+0,30	+1,00000" +0	+1,00000" +0		+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0
+0,20	+1,00000" +0	+1,00000" +0		+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0
+0,10	+1,00000" +0	+1,00000" +0		+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0
+0,00	+1,00000" +0	+1,00000" +0		+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0	+1,00000" +0

no convergence

convergence

Table 10. u3(x,y;ε) or u4(x,y;ε), ε = 0.001.

y \ x	-2,00	-1,00	+0,00	+1,00	+2,00	+3,00	+4,00	+5,00	+6,00
+2,00	+0,00000" +0	+0,00000" +0	+0,00000"-99	+3,21639"-81	+2,86277"-50	+6,19358"-36	+5,30266"-28	+4,84693"-23	+1,14212"-19
+1,90	+0,00000" +0	+0,00000" +0	+0,00000"-99	+4,33934"-68	+1,56164"-41	+1,30081"-29	+4,18933"-23	+4,61445"-19	+2,51300"-16
+1,80	+0,00000" +0	+0,00000" +0	+0,00000"-99	+9,13316"-56	+1,52203"-33	+6,82087"-24	+1,07402"-18	+1,72208"-15	+2,48873"-13
+1,70	+0,00000" +0	+0,00000"-99	+0,00012"-99	+2,37121"-44	+2,39439"-26	+8,54194"-19	+8,73574"-15	+2,48629"-12	+1,09985"-10
+1,60	+0,00000" +0	+0,00000"-99	+3,19904"-85	+5,81342"-34	+5,48296"-20	+2,44613"-14	+2,20506"-11	+1,37058" -9	+2,40936" -8
+1,50	+0,00000" +0	+0,00000"-99	+2,10170"-67	+9,94380"-25	+1,64667"-14	+1,53481"-10	+1,68969" -8	+3,22806" -7	+2,79407" -6
+1,40	+0,00000" +0	+0,00000"-99	+2,22759"-50	+8,43924"-17	+5,84102"-10	+2,02178" -7	+4,41749" -6	+3,56658" -5	+1,45399" -4
+1,30	+0,00000" +0	+0,00000"-99	+1,96526"-34	+2,42702"-10	+2,20058" -6	+6,24844" -5	+4,54189" -4	+1,52015" -3	+3,44072" -3
+1,20	+0,00000" +0	+0,00000"-99	+4,90932"-20	+1,54347" -5	+9,37887" -4	+5,70919" -3	+1,44587" -2	+2,56195" -2	+3,78592" -2
+1,10	+0,00000" +0	+0,00000"-99	+4,66719" -8	+1,61969" -2	+6,82093" -2	+1,14323" -1	+1,50337" -1	+1,78666" -1	+2,01489" -1
+1,00	+0,00000" +0	+0,00000"-99	+0,00000" +0	+6,38998" -1	+5,99135" -1	+5,81190" -1	+5,70424" -1	+5,63063" -1	+5,57662" -1
+0,90	+0,00000" +0	+0,00000"-99		+1,26180" +0	+1,13006" +0	+1,04806" +0	+9,90511" -1	+9,47460" -1	+9,13835" -1
+0,80	+0,00000" +0	+0,04559"-99		+1,27799" +0	+1,19733" +0	+1,15667" +0	+1,12639" +0	+1,10051" +0	+1,07746" +0
+0,70	+0,00000" +0	+4,09456"-82		+1,27800" +0	+1,19827" +0	+1,16232" +0	+1,14039" +0	+1,12461" +0	+1,11188" +0
+0,60	+0,00000" +0	+4,95340"-64		+1,27800" +0	+1,19827" +0	+1,16238" +0	+1,14084" +0	+1,12609" +0	+1,11518" +0
+0,50	+0,00000" +0	+3,63541"-47		+1,27800" +0	+1,19827" +0	+1,16238" +0	+1,14085" +0	+1,12612" +0	+1,11532" +0
+0,40	+0,00000" +0	+5,47846"-32		+1,27800" +0	+1,19827" +0	+1,16238" +0	+1,14085" +0	+1,12613" +0	+1,11532" +0
+0,30	+0,00000" +0	+4,27774"-19		+1,27800" +0	+1,19827" +0	+1,16238" +0	+1,14085" +0	+1,12613" +0	+1,11532" +0
+0,20	+0,00000" +0	+3,58372" -9		+1,27800" +0	+1,19827" +0	+1,16238" +0	+1,14085" +0	+1,12613" +0	+1,11532" +0
+0,10	+0,00000" +0	+6,95744" -3		+1,27800" +0	+1,19827" +0	+1,16238" +0	+1,14085" +0	+1,12613" +0	+1,11532" +0
+0,00	+0,00000" +0	+1,00000" +0		+1,27800" +0	+1,19827" +0	+1,16238" +0	+1,14085" +0	+1,12613" +0	+1,11532" +0

u3(x,y;ε)

u4(x,y;ε)

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