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AFDELING NUMERIEKE WISKUNDE (DEPARTMENT OF NUMERICAL MATHEMATICS) NW 162/83 DECEMBER

P.W. HEMKER

MIXED DEFECT CORRECTION ITERATION FOR THE SOLUTION OF A SINGULAR PERTURBATION PROBLEM

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Mixed defect correction iteration for the solution of a singular perturbation problem *)

by

P.W. Hemker

ABSTRACT

We describe a discretization method (mixed defect correction) for the solution of a two-dimensional elliptic singular perturbation problem. The method is an iterative process in which two basic discretization schemes are used: one with and one without artificial diffusion. The resulting method is stable and yields a 2nd order accurate approximation in the smooth parts of the solution, without using any special directional bias in the discretization. The method works well also for problems with interior or boundary layers.

KEY WORDS & PHRASES: *defect correction, singular perturbations, convection diffusion equation*

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*) This report will be submitted for publication elsewhere

1. INTRODUCTION

In recent years much research has been devoted to the numerical solution of Singular Perturbation Problems or what is called in the engineering community: the computation of convection dominated flows. The solutions of these problems are characterized by large regions where the solution is a smooth function of the independent variable and small regions (boundary - or interior layers) where the solution varies rapidly.

One of the first observations that are made for all known discretizations of these problems is that higher-order accurate schemes are strongly direction dependent, i.e. the difference scheme used or the Petrov-Galerkin weighting applied depends on the flow-direction in the equation. Symmetric schemes (finite differences or the usual Galerkin methods with symmetric weighting functions) are either not applicable (unstable) or only 1st order accurate.

On most feasible discretization grids it will be possible to represent properly the smooth parts of the solution but the grids are too coarse to fit the solution in the boundary layers. Therefore higher order accuracy is justifiably wanted only in the smooth parts. It makes no sense to require a high order polynomial approximation to the special layers, it is sufficient to locate them properly.

It is now known that Defect Correction yields the possibility to improve the order of accuracy of a stable low-order discretization by means of accurate but instable higher order methods [3]. Guided by this idea, in this paper we study whether it is possible to use a symmetric higher order scheme to improve the 1st order accurate solution obtained by a stable direction independent method. The purpose is to obtain a method in which no information is used about the flow direction and where still a high order of accuracy is obtained in the smooth parts of the solution. We shall see that the direct application of the defect correction principle does not satisfy our needs, but we can extend the defect correction idea and obtain a second order accurate discretization which has no directional bias.

In this paper results are collected that appeared in previous preliminary papers by the author on the same subject [6, 7, 8].

In the remaining part of this introduction we introduce the model problems that are studied and we briefly show some fundamental problems that arise with their numerical solution. In the 2nd section we describe the Local Mode Analysis that is used to study the solution methods. In the 3rd section we show how the direct application of the defect correction principle works out for our problems and in the next section the "mixed defect correction iteration" is introduced. The solution obtained by this method is analyzed in section 5 and in section 6 the convergence of the iteration process is studied. In the following section the solution is studied in the boundary layer and finally a few numerical examples are given.

As a model problem we study the singular perturbation equation

(1.1) $$L_\varepsilon u \equiv -\varepsilon \, \Delta u + \vec{a}.\nabla u = f,$$

in a two-dimensional region Ω. We refer to this equation as the convection-diffusion equation; $\vec{a}$ is the convection vector and $\varepsilon > 0$ is the diffusion parameter, which may be small compared to $|\vec{a}|$. This equation can be considered as a model equation for more complex real-life problems such as flows described by the Navier-Stokes equation, when the Reynolds number takes large values.

Allthough we study equation (1.1) with constant coefficients, we want to find numerical methods that are also applicable for variable $\vec{a}$; i.e. $\vec{a} = \vec{a}(x,y)$ or $\vec{a} = \vec{a}(x,y,u)$. In particular, we are interested in methods that are independent of the direction of $\vec{a}$ and independent of whether the grid is properly refined in possible boundary or interior layers, when ε is small.

As a simplification of the two-dimensional equation we also study the one-dimensional case. For this one-dimensional problem,

(1.2) $$L_\varepsilon u \equiv \varepsilon u_{xx} + 2u_x = f,$$

many numerical methods have already been investigated [9]. However, almost none of these methods are suitable for generalization in more dimensions.

An essential difficulty in the numerical solution of (1.1) with $0 < \varepsilon < h$, h the mesh-width, is the different type of approximation that is required in the smooth part of the solution and in the boundary or interior layers. In the smooth part an accurate approximation - possibly of high order - is desired, whereas for the boundary layer the proper location is most important, with the additional requirement that the effect of an (almost) discontinuity does not disturb the solution in the smooth parts.

For large values of ε the numerical solution of (1.1) or (1.2) gives no particular problems. Discretizations

$$(1.3) \qquad L_{h,\varepsilon} u_{h,\varepsilon} = f_h$$

are known for which $\| u_{h,\varepsilon} - u_\varepsilon \| = \mathcal{O}(h^2)$ as $h \to 0$, e.g. the usual central difference discretization. The errorbound remains valid for small values of ε:

$$\| u_{h,\varepsilon} - u_\varepsilon \| \le C_\varepsilon h^2 \quad \text{as} \quad h \le h_\varepsilon ,$$

but $C_\varepsilon \to \infty$ and $h_\varepsilon \to 0$ as $\varepsilon \to 0$. This means that the error estimate is of no use if we apply these discretizations with finite h and $\varepsilon \to 0$. In fact, for small ε, the usual discretizations may yield quite useless approximations. We show this by means of the 1-D model problem

$$(1.4) \qquad u_{xx} + 2u_x = 0, \quad x \in [0,\infty), \quad u(0) = 1, \quad u(\infty) = 0 .$$

Discretizing by central differences

$$(1.5) \qquad \varepsilon\, \Delta_+ \Delta_- u_h + (\Delta_+ + \Delta_-) u_h = 0,$$

we find

$$u_{h,\varepsilon}(jh) = \left(\frac{\varepsilon-h}{\varepsilon+h}\right)^j .$$

This is a second order approximation indeed; for jh fixed and $(\frac{h}{\varepsilon}) \to 0$

$$| u_{h,\varepsilon}(jh) - u_\varepsilon(jh) | = \left| \left(\frac{\varepsilon-h}{\varepsilon+h}\right)^j - (e^{-2h/\varepsilon})^j \right| \le C \left(\frac{h}{\varepsilon}\right)^2 ,$$

C independent of j, h and h/ε.
However, the solution of the reduced difference equation is

$$(1.6)\qquad u_{h,0}(jh) = \lim_{\varepsilon\to 0} u_{h,\varepsilon}(jh) = (-1)^j .$$

The influence of the boundary condition at $x = 0$ is significant over the whole domain of definition, whereas for the differential equation the influence of this boundary condition vanishes in the interior of the domain.

A well-known cure against this spurious influence of the boundary condition is "upwinding" or "artificial diffusion". In upwinding one-sided differences are used for the discretization of the first order term. In artificial diffusion, the diffusion constant ε is replaced by a larger value $\alpha = \varepsilon + \mathcal{O}(h)$. In both cases the spurious influence of the boundary layer far into the smooth part of the solution disappears at the expense of the fact that these discretizations are only accurate of order $\mathcal{O}(h)$. In the 1-D case "upwinding" is equivalent with "artificial diffusion" with $\alpha = \varepsilon + h|a|/2$.

The solution of the upwind discretization of (1.4)

$$(1.7)\qquad \varepsilon\, \Delta_+\Delta_- u_h + 2\Delta_+ u_h = 0$$

is

$$u_{h,\varepsilon}(jh) = \left(\frac{\varepsilon}{\varepsilon+2h}\right)^j .$$

In contrast with the central difference solution, we see that here the influence of the boundary condition vanishes in the interior of the domain as $\varepsilon \to 0$; but the discretization is only first order: for jh fixed and $(\frac{h}{\varepsilon}) \to 0$ we find

$$|u_{h,\varepsilon}(jh) - u_\varepsilon(jh)| \le C(\tfrac{h}{\varepsilon}).$$

2. LOCAL MODE ANALYSIS

We want to analyze separately the behaviour of the discretization (i) in the smooth parts of the solution, and (ii) in the boundary layers. Therefore we use local mode analysis, cf. Brandt [2] and Brandt and Dinar [1]. We consider equation (1.1) in two particular model problems:

(i) the inhomogeneous problem

$$(2.1) \qquad L_{h,\varepsilon} u_h = f_h$$

on a regular rectangular discretization of $\mathbb{R}^2$; u_h and f_h are ℓ_2-functions, and

(ii) the homogeneous problem

$$(2.2) \qquad L_{h,\varepsilon} u_h = 0$$

in a discretization of the half-space, of which the boundary is a grid-line; boundary conditions are given on this grid-line and u_h is bounded at infinity.

In both cases we consider the discretization of the constant coefficient problem on a regular rectangular grid and we decompse the solution in its Fourier modes ([5])

$$(2.3) \qquad u_h(jh) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int \hat{u}_h(\omega) e^{+i\omega hj} d\omega, \quad j \in \mathbb{Z}^2,$$

where $u_{h,\omega} = \hat{u}_h(\omega) e^{i\omega hj}$ is *the mode of frequency* ω in u_h; the amplitude of this mode with

$$\omega \in T_h^2 = \{\omega \mid \omega \in \mathbb{C}^2, \ \mathrm{Re}\, \omega_k \in [-\pi/h, \pi/h), k = 1,2\}$$

is given by

$$(2.4) \qquad \hat{u}_h(\omega) = \left(\frac{h}{\sqrt{2\pi}}\right)^2 \sum_j e^{-i\omega hj} u_h(jh).$$

If we consider the problem (2.1), the boundary condition imposes $\omega \in \mathbb{R}^2$; for (2.2) with Ω being the half-space, with boundary conditions at $x = 0$, we have $\mathrm{Im}\, \omega_1 \geq 0$, $\mathrm{Im}\, \omega_2 = 0$.

The modes being the eigenfunctions of the discrete operator L_h, we can define the *characteristic form* $\hat{L}_h(\omega)$ corresponding with the discrete operator L_h, by

$$(2.5) \qquad \widehat{L_h u_{h,\omega}} = \hat{L}_h(\omega)\hat{u}_{h,\omega}.$$

This characteristic form $\hat{L}_h(\omega)$ is the analogue of the *characteristic polynomial* or the *symbol* $\hat{L}(\omega)$ of the continuous operator L.
We now define consistency and stability of the operator L_h for each mode ω separately.

DEFINITION. The operator L_h is *consistent* with L *of order* p *for mode* $\omega \in T_h^2$ if

$$(2.6) \qquad |L_h(\omega) - \hat{L}(\omega)| \le C\, h^p \text{ for } h \to 0.$$

DEFINITION. The *stability of* L_h *for mode* $\omega \in T_h^2$ is the quantity $|\hat{L}_h(\omega)|$.

DEFINITION. The *local stability* of L_h, a discretization of L, *for* $\omega \in T_h^2 \cap \mathbb{R}^2$, $\hat{L}(\omega) \neq 0$ is

$$(2.7) \qquad |\hat{L}_h(\omega)|/|\hat{L}(\omega)|.$$

DEFINITION. The operator L_h is *locally stable* if

$$(2.8) \qquad \forall\rho > 0 \quad \exists\eta > 0 \quad \forall\omega \cap \mathbb{R}^2 \quad |\hat{L}(\omega)| > \rho \to |\hat{L}_h(\omega)|/|\hat{L}(\omega)| > \eta,$$

where $\eta = \eta(\rho)$ is independent of h.

DEFINITION. The operator $L_{h,\varepsilon}$, a discretization of L_ε, is *asymptotically stable* if

$$\forall\rho > 0 \quad \exists\eta > 0 \quad \forall\omega \in T_h^2 \cap \mathbb{R}^2 \quad \lim_{\varepsilon\to 0} |\hat{L}_\varepsilon(\omega)| > \rho \to \lim_{\varepsilon\to 0} \frac{|\hat{L}_{h,\varepsilon}(\omega)|}{|\hat{L}_\varepsilon(\omega)|} > \eta,$$

where $\eta = \eta(\rho)$ is independent of h.

DEFINITION. The operator $L_{h,\varepsilon}$ is ε*-uniformly stable* if (2.8) holds with $\eta = \eta(\rho)$ independent of h and ε.

To analyze the local behaviour of the discretization (1.5) of our one-dimensional problem we find its characteristic form

$$(2.9) \qquad \hat{L}_{h,\varepsilon}(\omega) = -4S(\varepsilon S - ihC)/h^2,$$

where $S = \sin(\omega h/2)$ and $C = \cos(\omega h/2)$.
Comparing this with the symbol $\hat{L}_\varepsilon(\omega) = -\varepsilon\omega^2 + 2i\omega$ of L_ε we find:
(1) the discretization (1.5) is consistent of order 2:

$$|\hat{L}_{h,\varepsilon}(\omega) - \hat{L}_\varepsilon(\omega)| \le C\, h^2\, |\varepsilon\omega^4 + i\omega^3| + \mathcal{O}(h^3);$$

(2) the discretization (1.5) is <u>not</u> asymptotically stable:

$$\lim_{\varepsilon\to 0} \hat{L}_{h,\varepsilon}(\pi/h) = 0, \quad \text{whereas} \quad \lim_{\varepsilon\to 0} \hat{L}_\varepsilon(\pi/h) = {}^{2\pi i}/h.$$

We find that $u_{h,\pi/h}$ is an unstable mode. This mode corresponds to

$$u_h(jh) = e^{i\pi j} = (-1)^j,$$

cf. eq. (1.6).

If we consider the discretization with artificial diffusion α, we find its characteristic form (2.9) with ε replaced by $\alpha > 0$. This discretization is
(1) consistent of order 1 if $|\alpha - \varepsilon| \le C_1 h$; viz.

$$(2.10) \qquad |\hat{L}_{h,\alpha}(\omega) - \hat{L}_\varepsilon(\omega)| \le C_1|\alpha - \varepsilon|\ |\omega|^2 + |\hat{L}_{h,\varepsilon}(\omega) - \hat{L}_\varepsilon(\omega)|$$
$$\le \mathcal{O}(|\alpha - \varepsilon|) + \mathcal{O}(h^2) = \mathcal{O}(h),$$

(2) locally stable, uniform in ε, if $|\alpha - \varepsilon| \ge C_2 h$; viz.

$$\left|\frac{\hat{L}_{h,\alpha}(\omega)}{\hat{L}_\varepsilon(\omega)}\right| = \left|\frac{\sin(\omega h/2)}{\omega h/2}\right| \left|\frac{\frac{2\alpha}{h}\sin(\omega h/2) - 2i\cos(\omega h/2)}{\varepsilon\omega - 2i}\right| \ge \frac{2\sqrt{2}}{\pi^2}\min(C_2, 1$$

These last two observations show that we obtain an ε-uniformly stable discretizati which is of order 1, only if we take $\alpha - \varepsilon = \mathcal{O}(h)$.

3. THE DEFECT CORRECTION PRINCIPLE

For the solution of linear problems, the defect correction is a general technique to approximately solve a "target" problem

(3.1) $$Lu = f$$

by means of an iteration process

(3.2) $$\tilde{L}u^{(i+1)} = \tilde{L}u^{(i)} - Lu^{(i)} + f, \quad i = 1,2,\dots .$$

The operator $\tilde{L}$, an approximation to L, is selected such that problems

$$\tilde{L}u^{(i+1)} = \tilde{f}.$$

with $\tilde{f}$ in a neighbourhood of f, are easy to solve. If $\tilde{L}$ is injective and the iteration process (3.2) converges to a fixed point $\tilde{u}$, then $\tilde{u}$ is clearly a solution of (3.1).

If two equations $\tilde{L}_h u_h = f_h$ and $L_h u_h = f_h$ are both discretizations of a problem $Lu = f$ (respectively consistent of order p and q, $p \le q$) and if $\tilde{L}_h$ satisfies the stability condition

(3.3) $$\|\tilde{L}_h^{-1}\| < C, \quad \text{uniform in } h,$$

then it is well known (cf. e.g. [3]) that, if the solution is sufficiently smooth, $u_h^{(i)}$ in the iterative process

(3.4.a) $$\begin{cases} \tilde{L}_h u_h^{(1)} = f_h, \\ \tilde{L}_h u_h^{(i+1)} = \tilde{L}_h u_h^{(i)} - L_h u_h^{(i)} + f_h, \end{cases}$$ (3.4.b)

satisfies

(3.4.c) $$\|u_h^{(i)} - R_h u\| = \mathcal{O}(h^{\min(q,ip)}),$$

where R_h denotes the restriction of u to the gridpoint values.

This error bound holds without a stability condition (3.3) for the accurate operator L_h.

Direct application of the defect correction principle to the solution of our singular perturbation problem suggest the application of (3.4) with $L_h = L_{h,\varepsilon}$, the 2nd order central difference discretization, and with $\tilde{L}_h = L_{h,\alpha}$, the artificial diffusion discretization. Then, the correction equation (3.4.b) has the simple form

$$(3.5) \qquad L_{h,\alpha} u_h^{(i+1)} = f_h + (\alpha - \varepsilon)\Delta_+\Delta_- u_h^{(i)} .$$

Since $L_{h,\alpha}$ is stable and consistent of order 1 and $L_{h,\varepsilon}$ is consistent of order 2, we obtain

$$(3.6) \qquad \| u_h^{(1)} - u \| = \mathcal{O}(h) \quad \text{and} \quad \| u_h^{(i)} - u \| = \mathcal{O}(h^2) \quad \text{for} \quad i > 1.$$

In the regions where $\Delta_+\Delta_- \, u_h^{(i)}$ is a good approximation to u_{xx}, (i.e. in the smooth part of the solution) $u_h^{(i+1)}$ is a better approximation to u than $u_h^{(1)}$. The error bounds (3.6), however, hold in the classical sense: for fixed ε and $h \to 0$. For a small ε/h and a general $i > 1$, the solution $u_h^{(i)}$ is not better than the central difference approximation, but in the first few iterands the instability of $L_{h,\varepsilon}$ has only a limited influence. This is shown in the following example.

For (1.4) we can compute the solutions in the defect correction process explicitly. Application of (3.5) with $\alpha = \varepsilon + h$ yields the solutions

$$u_h^{(1)}(jh) = \left(\frac{\varepsilon}{\varepsilon+2h}\right)^j ,$$

$$u_h^{(2)}(jh) = \left(\frac{\varepsilon}{\varepsilon+2h}\right)^j \left[1 - \frac{jh}{2} \cdot \frac{2h}{(\varepsilon+2h)}\right],$$

$$u_h^{(3)}(jh) = \left(\frac{\varepsilon}{\varepsilon+2h}\right)^j \left[1 - j \, \frac{2h^2}{(\varepsilon+2h)} \left\{1 - \frac{jh^2 - h(\varepsilon+h)}{(\varepsilon+2h)}\right\}\right].$$

The general solution is

$$u_h^{(m+1)}(jh) = \left(\frac{\varepsilon}{\varepsilon+2h}\right)^j \; p_m(j, h/\varepsilon),$$

where $p_m(j,h/\varepsilon)$ is a m-th degree polynomial in j, depending on the parameter h/ε. It is easily verified that, for ε fixed and $h \to 0$, the solutions are 2nd order accurate for $m = 1,2,\ldots$. For small values of ε/h, $p_m(j,h/\varepsilon)$ changes sign m times for $j = 0,1,2,\ldots, m+1$; i.e. in each iteration step of (3.5) one more oscillation appears in the numerical solution. The influence of the boundary condition at $x = 0$ vanishes in the interior after the first m+1 nodal points. Thus, we see that by each step of (3.4) the effect of the instability of $L_{h,\varepsilon}$ creeps over one meshpoint further into the numerical solution. Similar effects are found for the process in two dimensions.

The behaviour of the iterands $u_h^{(i)}$ can also be analyzed by local mode analysis. E.g. for the solution after one additional iteration step, $u_h^{(2)}$, we have

$$Q_{h,\varepsilon}u_h^{(2)} := L_{h,\alpha}(2L_{h,\alpha}-L_{h,\varepsilon})^{-1}L_{h,\alpha}u_h^{(2)} = f.$$

By Fourier analysis, analogous to (2.9), we find

$$\hat{Q}_{h,\varepsilon}(\omega) = \frac{-4S(\alpha S-ihC)^2}{h^2[(2\alpha-\varepsilon)S-ihC]},$$

from which we derive that $Q_{h,\varepsilon}$ is locally stable, uniformly for small ε:

$$|\hat{Q}_{h,\varepsilon}| \geq \frac{2}{\pi}\,|\omega|\,\frac{\min^2(1,\alpha/h)}{\max(1,2\alpha/h)}\,.$$

For all modes the operator $Q_{h,\varepsilon}$ is consistent of order two:

$$|\hat{Q}_{h,\varepsilon}-\hat{L}_{h,\varepsilon}| = \left|\frac{4S^3(\alpha-\varepsilon)^2}{h^2\{(2\alpha-\varepsilon)S-ihC\}}\right| = O(h^2).$$

We find $u_h^{(2)}$ to be a 2nd order accurate solution, uniformly in $\varepsilon > 0$, for the smooth components in the solution. The effect of improved accuracy in the smooth part of the solution, for small ε, is found in the actual computation indeed, see table 1.

$\varepsilon = 10^{-6}$	h = 1/10	ratio	h = 1/20	ratio	h = 1/40
	$\Vert\cdot\Vert = \max\vert y_i - y(x_i)\vert,\ i = 0,1,\ldots,N.$				
$\Vert y_h^{(1)} - R_h y\Vert$	0.3303	1.98	0.1665	2.00	0.0831
$\Vert y_h^{(2)} - R_h y\Vert$	0.6213	1.09	0.5714	1.06	0.5384
$\Vert y_h^{(3)} - R_h y\Vert$	0.7770	0.99	0.7791	1.01	0.7677
	$\Vert\cdot\Vert = \max\vert y_i - y(x_i)\vert,\ i = N/2, N/2+1,\ldots,N$				
$\Vert y_h^{(1)} - R_h y\Vert$	0.0698	2.38	0.02931	2.21	0.01326
$\Vert y_h^{(2)} - R_h y\Vert$	0.1037	3.83	0.02707	3.94	0.00687
$\Vert y_h^{(3)} - R_h y\Vert$	0.0544	4.58	0.01188	4.18	0.00284

Table 1. Errors in the numerical solution of $\varepsilon y'' + y' = f$ on $(0,1)$ by application of (3.4)-(3.5). Boundary conditions and f are such that $y(x) = \sin(4x) + \exp(-x/\varepsilon)$. Near the boundary at $x = 0$ the accuracy is only $\mathcal{O}(1)$. However, on a mesh with meshwidth h, the boundary layer cannot be represented anyway. For boundary layer resolution, locally a finer mesh is necessary. In the smooth part of the solution we find the order of accuracy as predicated by local mode analysis.

For the two-dimensional problem (1.1) we do *not* find this ε-uniform stability for $Q_{h,\varepsilon}$. Hence, with $\varepsilon \ll h$, it is not possible to find a 2nd order accurate approximation for (1.1) by application of a single step of (3.5). On the other hand, iterative application would result in the unwanted solution of the target-problem $L_{h,\varepsilon} u_h = f_h$.
In table 2 we show that, indeed, the error estimate (3.4.c) for a single step of (3.5), which holds for a fixed ε and $h \to 0$, does not hold uniformly in ε, not even in the smooth part of the solution.

	$h = 1/8$	ratio	$h = 1/16$	ratio	$h = 1/32$
$\varepsilon = 1$					
$\Vert u_h^{(1)} - R_h u \Vert$	0.0630	2.5	0.0255	1.7	0.0149
$\Vert u_h^{(2)} - R_h u \Vert$	0.0740	3.6	0.0203	4.0	0.00505
$\varepsilon = 10^{-6}$					
$\Vert u^{(1)} - R_h u \Vert$	0.790	1.4	0.578	1.5	0.380
$\Vert u_h^{(2)} - R_h u \Vert$	0.634	1.8	0.360	2.1	0.173

Table 2. The error is max-norm for (3.4), (3.5) with $\alpha = \varepsilon + h/2$, in the smooth part of the solution. The problem: $\varepsilon\Delta u + u_x = f$ on the unit square; with the Dirichlet boundary data and the data f such that $u(x,y) = \sin(\pi x)\sin(\pi y) + \cos(\pi x)\cos(3\pi y) + \dfrac{(\exp(-x/\varepsilon) - \exp(-1/\varepsilon))}{(1 - \exp(-1/\varepsilon))}$.

4. THE MIXED DEFECT CORRECTION PROCESS

In the previous section we considered the Defect Correction Process (3.4) in which in each iteration step an improved approximation is obtained to a (single) discrete target problem

$$L_h u_h = f_h, \quad L_h : X_h \to Y_h .$$

Now we consider the possibility of two different target problems

(P1) $$L_h^1 u_h^1 = f_h^1, \quad L_h^1 : X_h \to Y_h^1,$$

(P2) $$L_h^2 u_h^2 = f_h^2, \quad L_h^2 : X_h \to Y_h^2,$$

to be used in *one* iteration process, where both (P1) and (P2) are discretizations to the same problem

$$\text{(P)} \qquad Lu = f, \quad L: X \to Y.$$

To this end we introduce approximate inverse operators $\tilde{G}_h^1$ and $\tilde{G}_h^2$ to the operators L_h^1 and L_h^2 respectively (we assume $\tilde{G}_h^1$ and $\tilde{G}_h^2$ to be linear), and we define the Mixed Defect Correction Process (MDCP) by

$$\text{(4.1.a)} \quad \begin{cases} u_{i+\frac{1}{2}} = u_i - \tilde{G}_h^1(L_h^1 u_i - f_h^1), \\ u_{i+1} = u_{i+\frac{1}{2}} - \tilde{G}_h^2(L_h^2 u_{i+\frac{1}{2}} - f_h^2). \end{cases} \quad \text{(4.1.b)}$$

If $\tilde{G}_h^1$ and $\tilde{G}_h^2$ are invertible, we also introduce the notation $\tilde{L}_h^1 = (\tilde{G}_h^1)^{-1}$ and $\tilde{L}_h^2 = (\tilde{G}_h^2)^{-1}$ for the approximations to L_h^1 and L_h^2. The convergence of (4.1) is determined by the "amplification operator for the error"

$$\text{(4.2)} \qquad A_h = (I - \tilde{G}_h^2 L_h^2)(I - \tilde{G}_h^1 L_h^1).$$

By the fact that two different target operators L_h^1 and L_h^2 are used, it is clear that the sequence $u^{1/2}, u^1, u^{3/2}, u^2, \ldots$ generally does not converge. However, it is possible that limits

$$u_h^A = \lim_{i\to\infty} u_i \quad \text{and} \quad u_h^B = \lim_{i\to\infty} u_{i+\frac{1}{2}}, \qquad i = 1,2,\ldots,$$

exist. A stationary point u_h^A of (4.1) satisfies

$$\text{(4.3)} \qquad (I - A_h)u_h^A = (I - \tilde{G}_h^2 L_h^2)\tilde{G}_h^1 f_h^1 + \tilde{G}_h^2 f_h^2.$$

In the case that f_h^1 and f_h^2 can be written as $f_h^1 = \bar{R}_h^1 f$ and $f_h^2 = \bar{R}_h^2 f$ ($\bar{R}_h^1: Y \to Y_h^1$, $\bar{R}_h^2: Y \to Y_h^2$) equation (4.3) is equivalent with

$$\text{(4.4)} \qquad (\tilde{G}_h^1 L_h^1 + \tilde{G}_h^2 L_h^2 - \tilde{G}_h^2 L_h^2 \tilde{G}_h^1 L_h^1)u_h^A = (\tilde{G}_h^1 \bar{R}^1 + \tilde{G}_h^2 \bar{R}_h^2 - G_h^2 L_h^2 G_h^1 \bar{R}_h^1)f.$$

For u_h^A we prove the following theorem.

THEOREM. *Let* (P1) *and* (P2) *be two discretizations of* (P) *and let restrictions be defined by* $R_h: X \to X_h$, $\bar{R}_h^1: Y \to Y_h^1$, $\bar{R}_h^2: Y \to Y_h^2$.

(i) *Let* (P1) *and* (P2) *be such that* $f_h^1 = \bar{R}_h^1 f$, $f_h^2 = \bar{R}_h^2 f$.

(ii) *Let the local truncation error of* (P1) *and* (P2) *be of order* p_1 *and* p_2 *respectively.*

(iii) *Let* $\tilde{L}_h^k: X_h \to Y_h^k$, $k = 1,2$, *be stable discretizations of* L *and let* $\tilde{L}_h^k$ *be consistent with* L_h^k *of order* q_k.

(iv) *Let* $\|A_h\| \le C < 1$, *so that (4.1) converges for all* h *and let* u_h^A *be the stationary point of (4.1),*

then $\quad \|u_h^A - R_h u^*\| \le Ch^{\min(p_1+q_2,p_2)}$,

with u^* *the solution of* (P).

PROOF. From (iii) it follows that $\|\tilde{G}_h^k\| \le C$, $k = 1,2$, and $\|L_h^2 - \tilde{L}_h^2\| \le Ch^{q_2}$ uniformly in h. From (ii) follows for the truncation error $T_h^k = (L_h^k R - \bar{R}^k L)u^*$ that $\|T_h^k\| = O(h^{p_k})$.
From (iv) we know $\|A_h\| \le C < 1$ and hence $\|(I-A_h)^{-1}\| \le 1/(1-C)$.
Now we see from (4.3)

$$(I-A_h)(R_h u^* - u_h^A) = (I-\tilde{G}_h^2 L_h^2)\tilde{G}_h^1(L_h^1 R_h u^* - f_h^1) - \tilde{G}_h^2(L_h^2 R_h u^* - f_h^2) =$$
$$= \tilde{G}_h^2(\tilde{L}_h^2 - L_h)\tilde{G}_h^1 T_h^1 - \tilde{G}_h^2 T_h^2,$$

and hence

$$\|R_h u^* - u_h^A\| \le \|(I-A_h)^{-1}\|\|\tilde{G}_h^2\|\{\|\tilde{L}_h^2 - L_h\|\|\tilde{G}_h^1\|\|T_h^1\| + \|T_h^2\|\}$$
$$\le C(Ch^{q_2}.Ch^{p_1} + Ch^{p_2})$$
$$\le Ch^{\min(p_1+q_2,p_2)}. \qquad \text{Q.E.D.}$$

Similarly we find for u_h^B

$$\|R_h u^* - u_h^B\| \le Ch^{\min(p_2+q_1,p_1)}.$$

For the singular perturbation problem (1.1) we take

(4.5)
a) $L_h^1 = L_{h,\varepsilon}$ the central difference (or FEM) discrete operator,
b) $L_h^2 = \tilde{L}_h^1 = L_{h,\alpha}$ the artificial diffusion discrete operator, and
c) $\tilde{L}_h^2 = D_{h,\alpha} := 2\ \mathrm{diag}(L_{h,\alpha})$.

By this choice, (4.1a) is a defect correction step towards the 2nd order accurate solution of $L_{h,\varepsilon}u_h = f_h$, by means of the operator $L_{h,\alpha}$. The second step (4.1.b) is only a damped Jacobi-relaxation step towards the solution of the problem $L_{h,\alpha}u_h = f_h$. For this choice of operators, the above theorem yields, for a fixed ε, the error bounds

(4.6) $$\|R_h u_\varepsilon - u_{h,\varepsilon}^A\| \le C_\varepsilon h \text{ and } \|R_h u_\varepsilon - u_{h,\varepsilon}^B\| \le C_\varepsilon h^2,$$

where u_ε is the exact solution. The defect correction step (4.1.a) generates a 2nd order accurate solution and may introduce high-frequency unstable components. The damped Jacobi relaxation step (4.1.b) is able to reduce the high-frequency errors.

In this paper we shall mainly be concerned with the convergence of the iteration process (4.1)-(4.5) and with the properties of its fixed points, the *"the stationary solutions"*. These solutions u_h^A and u_h^B can be characterized as solutions of linear systems

(4.7) $$[L_h^1 + L_h^2(\tilde{L}_h^2)^{-1}(L_h^2 - L_h^1)]u_h^A = f_h,$$

and

(4.8) $$[L_h + (L_h^2 - L_h^1)(\tilde{L}_h^2)^{-1}L_h^2]u_h^B = [I + (L_h^2 - L_h^1)(\tilde{L}_h^2)^{-1}]f_h,$$

with L_h^1, L_h^2 and $\tilde{L}_h^2$ as in (4.5).
In short, we denote eq. (4.7) as

(4.9) $$M_{h,\varepsilon}u_h^A = f_h.$$

The method described here is to a large extent similar to the double discretization method of Brandt [2]. In that method a multiple grid iteration process is used for the solution of (1.1). The relaxation method in each MG-cycle is taken with a stable "target" equation, and the course grid correction is made by means of a residual that is computed with respect to another (accurate) equation. In the double discretization method this is applied to all levels of discretization, to obtain efficiently an approximation of the continuous equation. There, however, it is hard to characterize the solution finally obtained. In our MDCP method we use also two target equations, but we restrict the treatment to a single level of discretization. We don't specify the way by which the stable linear system (4.1.a) is solved and, thus, we can characterize the two solutions obtained by (4.7) and (4.8).

5. LOCAL MODE ANALYSIS OF THE MDCP SOLUTION

The characteristic forms of the different discretizations of the one dimensional model problem (1.2) are, for central differencing, upwinding, and the MDCP discretization respectively:

$$(5.1) \qquad \hat{L}_{h,\varepsilon}(\omega) = -\frac{4\varepsilon}{h^2} S^2 + \frac{4i}{h} SC,$$

$$(5.2) \qquad \hat{L}_{h,\alpha}(\omega) = -\frac{4\varepsilon}{h^2} S^2[1+\frac{h}{\varepsilon}] + \frac{4i}{h} SC,$$

$$(5.3) \qquad \hat{M}_{h,\varepsilon}(\omega) = -\frac{4\varepsilon}{h^2} S^2[1+\frac{h}{\varepsilon} S^2] + \frac{4i}{h} SC[1+\frac{h}{\varepsilon+h} S^2],$$

where $S = \sin(\omega h/2)$ and $C = \cos(\omega h/2)$.

<u>THEOREM</u>. *The operator* $M_{h,\varepsilon}$ *defined by the* MDCP *process* (4.1)-(4.5) *applied to the model equation* (1.2) *is consistent of* 2nd *order and* ε-*uniformly stable.*

<u>PROOF</u>. See [6].

From this result, obtained by local mode analysis, we expect that u_h^A shows 2nd order accuracy in the smooth part of the solution. This is found in the actual computation indeed. Results are shown in table 3.

	h = 1/10	ratio	h = 1/20	ratio	h = 1/40
	$\lVert . \rVert = \max \lvert y_i - y(x_i) \rvert, \quad i = 0,1,\dots,N$				
$\lVert y_h^A - R_h y \rVert$	0.208	0.92	0.227	0.97	0.233
$\lVert y_h^B - R_h y \rVert$	0.565	0.94	0.604	0.98	0.614
	$\lVert . \rVert = \max \lvert y_i - y(x_i) \rvert, \quad i = N/2, N/2+1,\dots,N$				
$\lVert y_h^A - R_h y \rVert$	0.02507	3.83	0.00653	3.96	0.00165
$\lVert y_h^B - R_h y \rVert$	0.05953	3.83	0.01556	3.97	0.00392

Table 3. Errors in the numerical solution by MDCP; the same problem has been solved as for table 1.

An analysis similar to the one-dimensional case, can be given for the two-dimensional model problem (1.1).
The corresponding difference operator is given by

$$(5.4) \qquad L_{h,\varepsilon} \equiv \frac{-\varepsilon}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} + \frac{a_1}{(4+2p)h}\begin{bmatrix} & -p & +p \\ -2 & 0 & 2 \\ -p & +p & \end{bmatrix} + \frac{a_2}{(4+2p)h}\begin{bmatrix} & 2 & p \\ p & 0 & -p \\ -p & -2 & \end{bmatrix}.$$

With $p = 0$ it corresponds to the central difference discretization; with $p = 1$ it describes the FEM discretization on a regular triangulation with piecewise linear trial- and test-functions. The discretization operator is used either with the given diffusion coefficient ε or with this coefficient replaced by an artificially enlarged diffusion coefficient $\alpha = \varepsilon + Ch$, where C is independent of ε and h. Also for the 2-D equation we define the MDCP by (4.1)-(4.5). For the two-dimensional problem (1.1) the characteristic form of the discrete operators is

$$(5.5) \qquad \hat{L}_{h,\varepsilon}(\omega) = \frac{-4\varepsilon}{h^2} S^2 + \frac{4i}{h} T, \quad \hat{D}_{h,\alpha}(\omega) = \frac{-8\alpha}{h^2},$$

and

$$(5.6)\qquad \hat{M}_{h,\varepsilon}(\omega) = \frac{-4\varepsilon}{h^2} S^2[1 + \frac{\alpha-\varepsilon}{2\varepsilon} S^2] + \frac{4i}{h} T[1 + \frac{\alpha-\varepsilon}{2\alpha} S^2],$$

where

$$T = -[a_1 S_\phi(2C_\phi + pC_{\phi+2\theta}) + a_2 S_\theta(2C_\theta + pC_{\theta+2\phi})]/(4+2p),$$

$$S^2 = S_\phi^2 + S_\theta^2, \quad S_\phi = \sin\phi, \quad C_\phi = \cos\phi,$$

$$\phi = \omega_1 h/2 \quad \text{and} \quad \theta = \omega_2 h/2.$$

For $\varepsilon = 0$ the continuous operator L_ε is unstable for the modes $u_\omega = e^{i\omega x}$ with frequencies $\omega = (\omega_1,\omega_2)$ that are perpendicular to $a = (a_1,a_2)$. For $\varepsilon = 0$ the discrete operator $L_{h,\varepsilon}$ is unstable for the modes $u_{h,\omega} = e^{i\omega h}$ for which ω satisfies $T(\omega) = 0$. In the finite difference discretization (p=0), these modes $\omega = (\omega_1,\omega_2)$ are simply characterized by

$$a_1 \sin(\omega_1 h) + a_2 \sin(\omega_2 h) = 0.$$

The operator $L_{h,\alpha}$ has no unstable modes for $\varepsilon \to 0$ and it is consistent (of order one) with $L_{\varepsilon,h}$ if and only if $|\alpha-\varepsilon| = \mathcal{O}(h)$ as $h \to 0$. The 2nd order consistency of $M_{h,\varepsilon}$ and its asymptotic stability are proved similarly to the one-dimensional case.

<u>THEOREM</u>. *The operator* $M_{h,\varepsilon}$, *defined by the process* (4.1)-(4.5), *applied to the model equation* (1.1) *with central difference or finite element discretization for* $L_{h,\varepsilon}$ *and with artificial diffusion,* $\alpha = \varepsilon + Ch$, *is consistent of 2nd order and is asymptotically stable.*

<u>PROOF</u>. See [7].

As for the one-dimensional case we expect from this result 2nd order accuracy in the smooth part of the solution. Results for an actual computation are shown in table 4. In contrast with the direct defect correction method as treated in section 3, we see here that a 2nd order accurate solution is obtained also for small ε indeed.

	h = 1/8	ratio	h = 1/16	ratio	h = 1/32
$\varepsilon = 1$					
$\Vert u_h^A - R_h u \Vert$	0.0693	3.5	0.0201	3.9	0.00516
$\Vert u_h^B - R_h u \Vert$	0.0780	3.6	0.0214	4.0	0.00533
$\varepsilon = 10^{-6}$					
$\Vert u_h^A - R_h u \Vert$	0.459	3.4	0.132	4.5	0.0291
$\Vert u_h^B - R_h u \Vert$	0.608	3.8	0.159	4.7	0.0335

Table 4. The error is the max-norm for u_h^A and u_h^B measured in the smooth part of the solution.
The problem solved is the same as used for table 2.

Computing two final solutions u_h^A and u_h^B, we are interested to know what the difference between both solutions is. From (4.1)-(4.5) we easily derive

$$(5.7) \qquad u_h^B - u_h^A = (\alpha-\varepsilon)D_{h,\alpha}^{-1}\Delta_+\Delta_-\, u_h^A.$$

From this formula we see that $u_h^B - u_h^A$ is large where $\alpha - \varepsilon$ and the 2nd order differences of u_h^A are large. Hence, this is the region where the influence of the artificial diffusion is significant. From (5.7) we immediately derive

$$\widehat{u_h^B - u_h^A} = \frac{\alpha-\varepsilon}{2\alpha} S^2\, \hat{u}_h^A,$$

from which we conclude (cf. [8]) that for low frequences (where $S \approx h$)

$$\widehat{u_h^B - u_h^A} = \mathcal{O}(h^2) \quad \text{for} \quad h \to 0,$$

uniformly for all ε. For the high frequencies (where $S \approx 1$): for fixed ε we find

$$u_h^B - u_h^A = \mathcal{O}(h) \quad \text{for} \quad h \to 0$$

and for $0 \le \varepsilon < h$ with $h \to 0$

$$u_h^B - u_h^A = \mathcal{O}(1).$$

6. THE CONVERGENCE OF MDCP ITERATION

In this section we consider the rate of convergence of the process (4.1)-(4.5). By local mode analysis we show at what rate the different frequencies in the error are damped. The amplification operator for the error, A_h, is given by (4.2). Its characteristic form is

$$\hat{A}_h(\omega) = \frac{(\alpha-\varepsilon)\hat{\Delta}_h(\omega)}{\hat{L}_{h,\alpha}(\omega)} \cdot \frac{\hat{L}_{h,\alpha}(\omega)-\hat{D}_{h,\alpha}}{\hat{D}_{h,\alpha}} .$$

Using this expression for the *one-dimensional* model problem we find

$$(6.1) \qquad \hat{A}_h(\omega) = \frac{(\alpha-\varepsilon)SC}{-\alpha} \; \frac{[SC(\alpha^2-h^2)+ih\alpha]}{\alpha^2S^2+h^2C^2} ,$$

and

$$(6.2) \qquad |\hat{A}_h(\omega)| \le \frac{\alpha-\varepsilon}{\alpha} \; \tfrac{1}{2}\sqrt{\frac{1}{4}\left(\frac{\alpha}{h}-\frac{h}{\alpha}\right)^2 + \max{}^2\left(\frac{h}{\alpha},\frac{\alpha}{h}\right)} .$$

With the upwinding amount of artificial viscosity, $\alpha = \varepsilon + h$, we derive from (6.2) that $|\hat{A}_h(\omega)| \le \tfrac{1}{2}\sqrt{2}$, i.e. the process converges with a finite rate for all frequencies. Such a simple result is not obtained in the two-dimensional case.

For the *two-dimensional* problem we find

$$(6.3) \qquad |\hat{A}_h(\omega)| = \frac{(\alpha-\varepsilon)S^2\sqrt{\left(\frac{\alpha}{h}C^2S^2-\frac{h}{\alpha}T^2\right)^2+4T^2}}{2h\left[\left(\frac{\alpha}{h}S^2\right)^2+T^2\right]} ,$$

where $C^2 = C_\phi^2 + C_\theta^2$. It is easy to show that $|\hat{A}_h(\omega)| \le 1$ for all ω. However, for some frequencies convergence is slow. E.G. for the unstable modes ω of $L_{h,\varepsilon}$, for which $T(\omega) = 0$, we find

$$|\hat{A}_h(\omega)| = \frac{\alpha-\varepsilon}{2\alpha}\,(C_\phi^2+C_\theta^2),$$

i.e. for small ε, along $T(\omega) = 0$, in the neighbourhood of $\omega = 0$, convergence is slow. If we set $\alpha = \varepsilon + \gamma h$, then, considering the limit for $\varepsilon \to 0$, we obtain

$$(6.4)\qquad |\hat{A}_h(\omega)| = \left|\frac{\gamma S^2}{T}\right|\left(1+\frac{1}{4}\left(\frac{T}{\gamma} - C^2\left(\frac{\gamma S^2}{T}\right)\right)^2\right)^{\frac{1}{2}}\left(1+\left(\frac{\gamma S^2}{T}\right)^2\right)^{-1}.$$

To understand this expression, we introduce a new coordinate system in the frequency space. We define lines with constant $y = \gamma S^2/T$ and lines with constant $t = T/2\gamma$. Then

$$(6.5)\qquad |\hat{A}_h(\omega)| = \frac{|y|}{1+y^2}\,\sqrt{1+(t-(1-yt)y)^2}.$$

In the neighbourhood of the origin, lines of constant y are approximately circles tangent in the origin to the line $T(\omega) = 0$, the value of y is proportional to the radius. Lines of constant t are lines approximately parallel to the line $a_1\omega_1 + a_2\omega_2 = 0$, t is proportional to the distance to this line. We see that for small y

$$|\hat{A}_h(\omega)| \approx y\sqrt{1+t^2} = \mathcal{O}(y);$$

and for large y, i.e. in the neighbourhood of $t = 0$,

$$|\hat{A}_h(\omega)| \approx \sqrt{y^{-2}+\frac{1}{4}C^2} \approx \frac{1}{2}(C_\phi^2+C_\theta^2).$$

Thus we see that low frequencies converge fast along the convection direction $\vec{a}$ and that convergence is slow (only!) in the direction perpendicular to the convection direction (i.e. for those ω with $T(\omega) = 0$.)

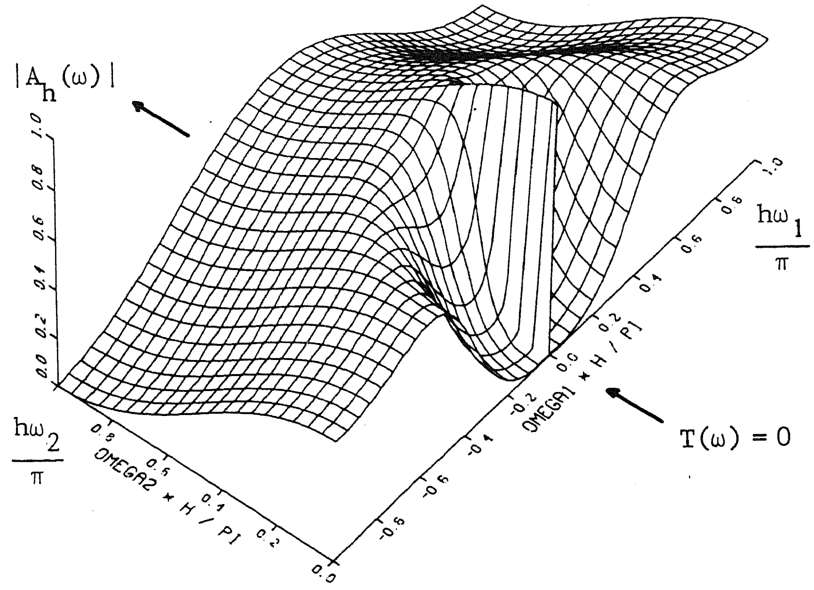

Figure 1. The MDCP convergence rate for the equation $-\varepsilon\Delta u + u_x = f$; discretization by finite differences.

7. BOUNDARY ANALYSIS OF THE MDCP SOLUTIONS

In section 5, by Local Mode Analysis for ℓ_2-functions on an unbounded domain, we saw that the MDCP discretization is asymptotically stable with respect to the right-hand side f_h. To analyze the effects of the boundary data, we consider the homogeneous problem (2.2) in a discretization of the right half-space ($x \geq 0$), the boundary $x = 0$ being a grid-line. Dirichlet boundary data are given on this boundary and we consider solutions that are bounded at infinity. This situation is again studied by mode analysis. Now we use complex modes, $\omega = (\omega_1, \omega_2) \in \mathbb{C}^2$; $\omega_2 \in \mathbb{R}$ is given by the boundary data and $\hat{L}_h(\omega) = 0$ is solved for $\omega_1 \in \mathbb{C}$. Those solutions ω_1 for which $\mathrm{Im}\ \omega_1 \geq 0$ determine the behaviour of the discretization near the boundary at $x = 0$.

In this way, we first treat the *one-dimensional* model problem (1.2) with $\alpha = \varepsilon + h$. For this problem the only possible inhomogeneous boundary data are $u_h(0) = 1$. The modes $u_{h,\omega}(jh) = e^{i\omega hj} = \lambda^j$ for which the homogeneous equation (4.9) is satisfied, are determined by

$$\hat{M}_{h,\varepsilon}(\omega) = 0. \tag{7.1}$$

This is a 4th degree polynomial in λ. With $\varepsilon = 0$ we find for (7.1) the solutions $\lambda = 1$, $\lambda = 0$, $\lambda = 2 \pm \sqrt{5}$. From (5.3) it is clear that for all $\varepsilon/h > 0$, $\lambda = 1$ is a solution and no other solutions with $|\lambda| = 1$ exist. Since all λ are continuous functions of ε, we have for all $\varepsilon \geq 0$ two λ's with $|\lambda| < 1$ and one λ with $|\lambda| > 1$. The two λ's with $|\lambda| < 1$ determine the behaviour of the solution near the boundary at $x = 0$. For small values of ε/h we find $\lambda_1 = \mathcal{O}(\varepsilon/h)$ and $\lambda_2 = 2 - \sqrt{5} + \mathcal{O}(\varepsilon/h)$. These values show that in the numerical boundary layer, for small ε/h, the influence of the boundary data decreases with a fixed rate per meshpoint. I.e. the width of the numerical boundary layer is only $\mathcal{O}(h)$.

Only λ_1 and λ_2 determine what modes appear in the solution of

$$M_{h,\varepsilon} u_h = 0, \quad u_h(0) = 1;$$

The difference operator at the meshpoint next to the boundary determines what linear combination of λ_1^j and λ_2^j forms u_h^A and u_h^B. A more detailed computation shows

$$\text{(7.2)} \quad \begin{aligned} u_h^A(jh) &= -(2+2\sqrt{5})\frac{\varepsilon}{h}\,\lambda_1^j + [1 + (2+2\sqrt{5})\frac{\varepsilon}{h}]\lambda_2^j + \mathcal{O}((\frac{\varepsilon}{h})^2), \\ u_h^B(jh) &= -(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5})\lambda_1^j + (\frac{3}{2}+\frac{1}{2}\sqrt{5})\lambda_2^j + \mathcal{O}((\frac{\varepsilon}{h})). \end{aligned}$$

This describes completely the behaviour of the 1-D numerical boundary layer solution.

We analyze the two-dimensional model problem in a similar way. For given boundary data

$$u_{h,\omega}(jh) = \exp(i\omega_2 h_2 j_2) \quad \text{for} \quad j_1 = 0,$$

we compute the modes

$$u_{h,\omega}(jh) = e^{i\omega hj} = e^{i\omega_1 h_1 j_1}\, e^{i\omega_2 h_2 j_2} = \lambda^{j_1}\, e^{i\omega_2 h_2 j_2}$$

that satisfy $M_{h,\varepsilon} u_{h,\omega} = 0$ for $j_1 > 0$, and we determine the corresponding $|\lambda|$.

To simplify the computation, we restrict ourselves to the finite difference star (i.e. $p = 0$ in eq. (5.4)) and artificial diffusion

$\alpha = \varepsilon + h|a_1|/2$, $a_1 \neq 0$. First we consider boundary data with $\omega_2 = 0$. For $\varepsilon = 0$ we determine λ from

$$(7.3) \qquad \hat{M}_{h,0}(\omega) = \frac{-2\alpha}{h^2} S^4 + \frac{2i}{h} T[2+S^2] = 0.$$

We find the solutions $\lambda_0 = 0$, $\lambda_1 = 1$, $\lambda_{2,3} = 3 \pm \sqrt{12}$. Next we consider $\omega_2 \neq 0$. From (7.3) it follows that no real $\omega \in [-\pi/h,\pi/h]^2$, $\omega \neq (0,0)$, exists such that $\hat{M}_{h,0}(\omega) = 0$. Hence, except for $\omega_2 = 0$, no λ exists with $|\lambda| = 1$. All λ's are continuous functions of ω_2 and for small ω_2 we know

$$\lambda_1 = 1 - i\frac{a_2}{a_1}\omega_2 h + \mathcal{O}((\omega_2 h)^4).$$

Hence $|\lambda_1| \geq 1$ and for all $\omega_2 \in [-\pi/h,\pi/h]$ there are two λ's with $|\lambda| < 1$ and two λ's with $|\lambda| \geq 1$. The two small λ's, considered as functions of $\omega_2 \in [-\pi/h,\pi/h]$ describe a curve inside the unit circle in $\mathbb{C}$. The curves are closed subsets of $\mathbb{C}$ and have no point in common with the unit-circle. Thus, we see that $C = \max_{\omega_2} |\lambda_{\omega_2}|$ exists and $|\lambda| \leq C < 1$. Thus we see that for all ω_2 the small eigenvalues are separated from 1.
To study the case $\varepsilon \neq 0$, we consider $\hat{M}_{h,\varepsilon}$ instead of $\hat{M}_{h,0}$ and we see again that for bounded ε/h no real ω exist for $\hat{M}_{h,\varepsilon}(\omega) = 0$ except $\omega = (0,0)$, which yields a single root $\lambda_1 = 1$ for all ε/h. Since all λ's are continuous functions of ε/h we conclude that for all ω_2 and all $0 \leq \varepsilon/h < C$ there are two λ's with $|\lambda| \geq 1$ and two λ's with $|\lambda| \leq C_2 < 1$. We conclude that, also in the two-dimensional case for small ε/h, the influence of the boundary data decreases with a fixed rate per meshpoint, i.e. the width of the numerical boundary layer is $\mathcal{O}(h)$.

This non-constructive proof for the existence of $\max|\lambda(\omega_2)| < 1$ allows the possibility of a large $|\lambda| < 1$, such that the existence may be of little practical use. In the numerical examples we see that the numerical boundary layer extends only over a few meshlines in the neighbourhood of the boundary indeed.

8 NUMERICAL EXAMPLES

In this section we show some numerical results obtained for the model problem, in the presence of boundary or interior layers. We include also an example with variable coefficients.

8.1 In the first example we show the solution of problem (1.1) with Dirichlet boundary conditions on the unit square; $\varepsilon = 10^{-6}$, $\vec{a} = (-1,0)$, the function f and the boundary data are chosen such that $u(x,y) = (M\exp(-x/\varepsilon)-1)/(M-1)$ with $M = \exp(1/\varepsilon)$. The problem is discretized with the standard FEM with piecewise linear functions on a regular triangulation. The mesh was chosen with h = 1/8, 1/16, 1/32. With $\alpha = \varepsilon + h/2$ the numerical solution u_h^A is shown in figure 2. We see that the numerical boundary layer has width $\mathcal{O}(h)$. Only a few meshlines near the boundary layer are affected by the downstream Dirichlet boundary condition.

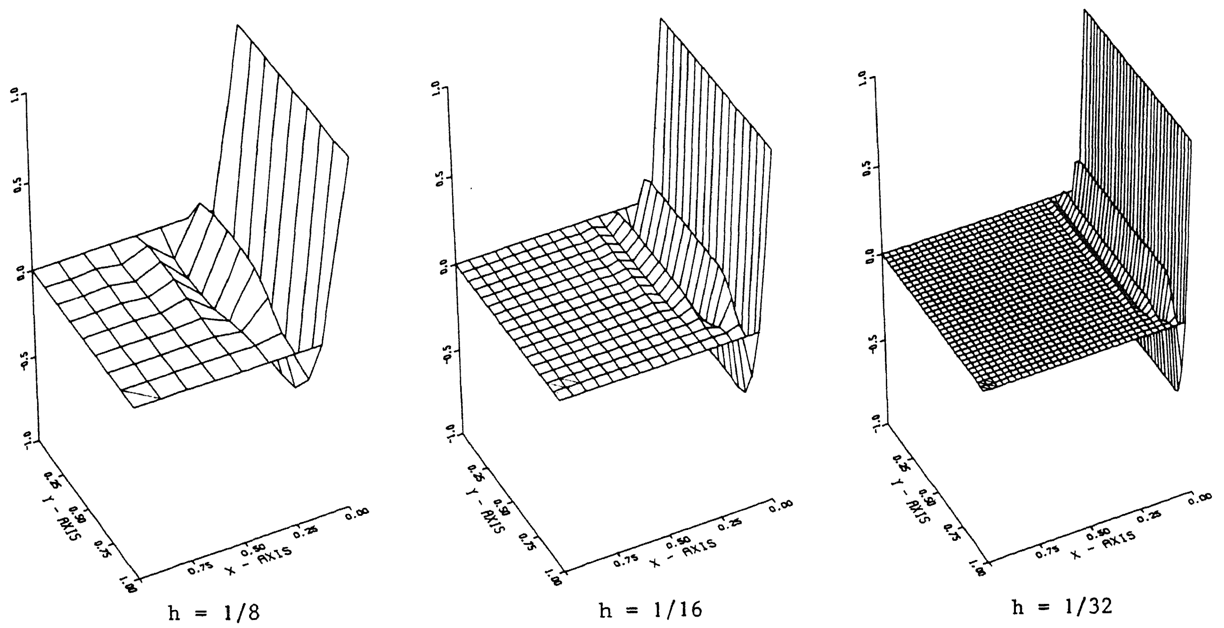

Figure 2. Example 8.1

8.2 In the second example we show again solutions of (1.1) with $\varepsilon = 10^{-6}$, $\vec{a} = (1,0)$. Dirichlet BCs are given, except at the outflow boundary, where natural BCs were used. The rhs and the Dirichlet boundary data were chosen such that

$$u(x,y) = -\sqrt{-x_0/(x-x_0)}\,\exp(-(y-y_0)^2/(4\varepsilon(x-x_0))).$$

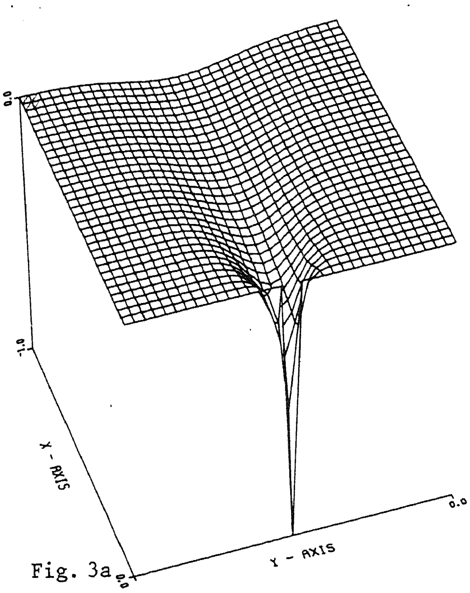

Fig. 3a

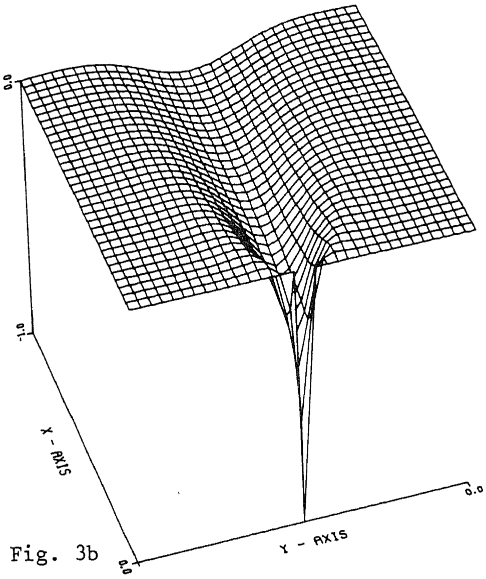

Fig. 3b

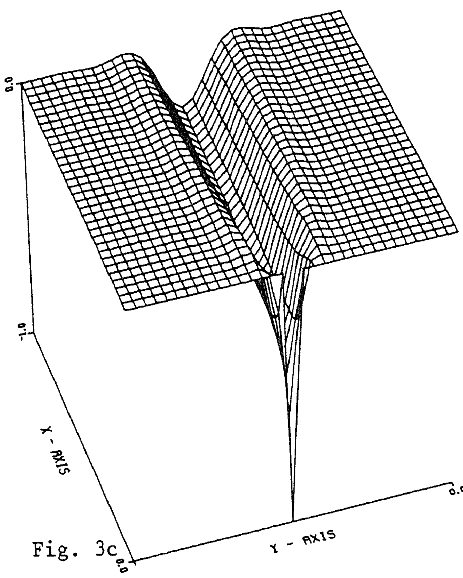

Fig. 3c

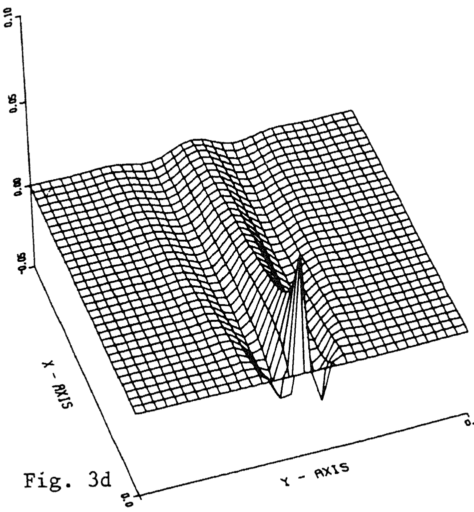

Fig. 3d

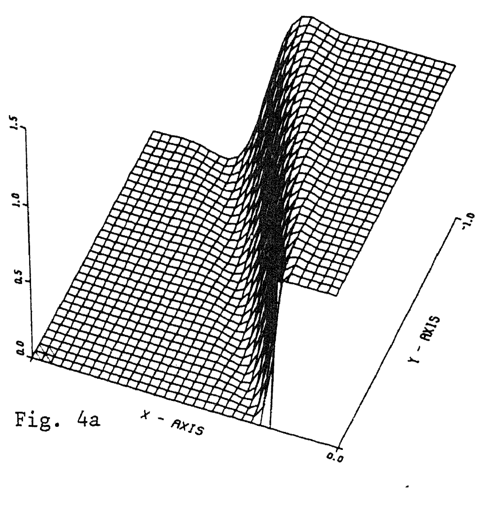

Fig. 4a

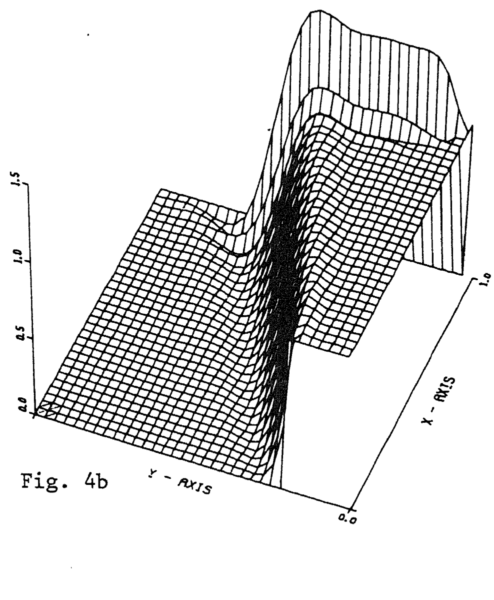

Fig. 4b

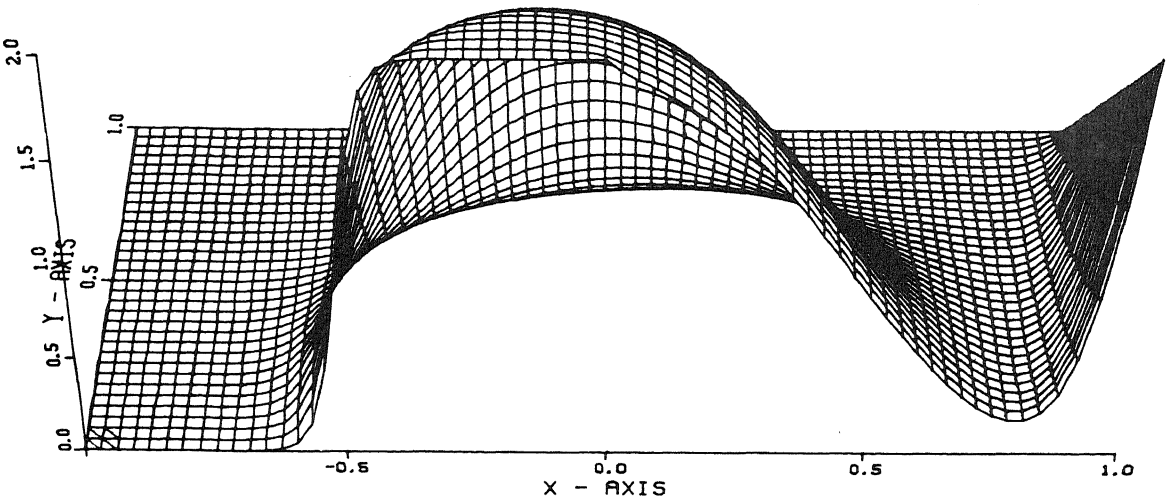

Figure 5a. Example 8.4 with artificial diffusion.

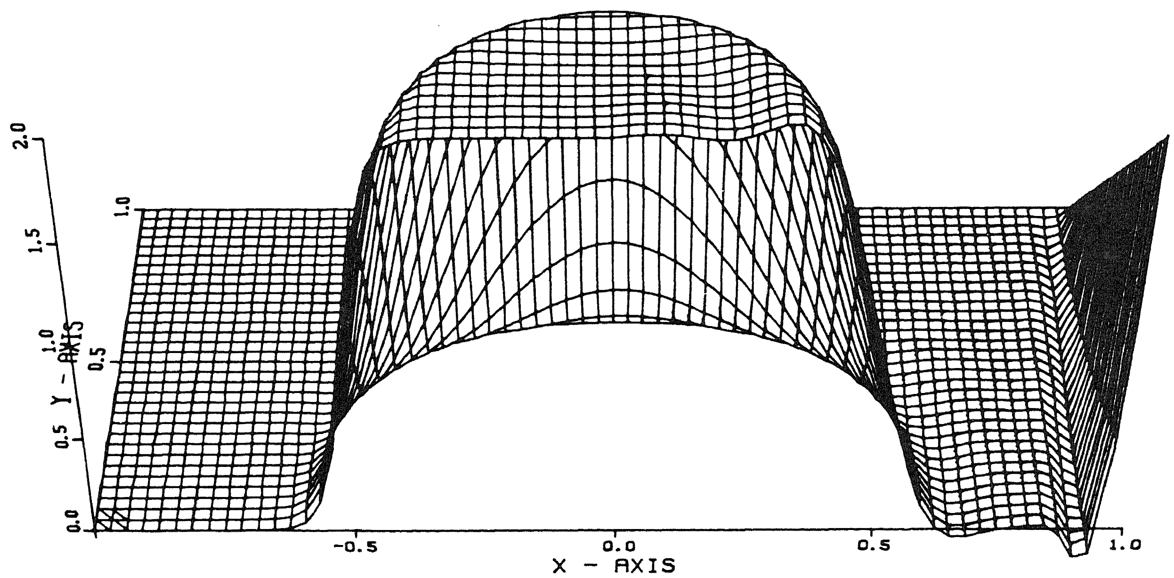

Figure 5b. Example 8.4, solution u_h^A.

with $x_0 = -0.1$ and $y_0 = 0.5$. These data cause a strong parabolic interior layer in the solution. (The solution $u(x,y)$ also satisfies the homogeneous equation $-\varepsilon u_{yy} + u_x = 0$.) In figure 3 we see the numerical solution of this problem (a) by application of artificial diffusion with $\alpha = \varepsilon + h/2$, (b) by application of a single defect correction step, (c) the solution u_h^A and (d) the difference between u_h^A and u_h^B. We see that the solutions u_h^A and u_h^B yield much sharper layers than (a) or (b). Further, we see that $u_h^B - u_h^A$ is large where the influence of the singular perturbation is significant (see eq. (5.7)).

8.3 In the 3rd example we show the solution of (1.1) with $\varepsilon = 10^{-6}$, $\alpha = \varepsilon + h/2$, $a = (\cos(\phi), \sin(\phi))$, $\phi = 22\frac{1}{2}^{\circ}$. Dirichlet BCs are given at the inflow boundary; $u = 0$ or $u = 1$ with a discontinuity at $(0,3/16)$, so that an interior layer is created. At the outflow boundary homogeneous Neumann (fig. 4a) or Dirichlet (fig. 4b) boundary data are given. We see that also for a skew flow a rather sharp profile is found (cf. [10]). The boundary layer at the outflow Dirichlet boundary shows the same behaviour as in example 1. Similar behaviour of the solutions is found for other angles ϕ (cf. [8]).

8.4 In this last example we use a problem with variable coefficients: equation (1.1) on a rectangle $[-1,+1] \times [0,1]$ with $\varepsilon = 10^{-6}$, $a = (y(1-x^2), -x(1-y^2))$. This represents a flow around the point $(0,0)$, with inflow boundary $-1 \le x \le 0$, $y = 0$ and outflow at $0 < x \le 1$, $y = 0$. Dirichlet boundary conditions are given at all boundaries except the outflow boundary where Neumann boundary conditions were used. At the inflow boundary a flow profile is given: $u(x,0) = 1 + \tanh(10+20x)$. This results in an interior layer. For the boundary condition at $(x=1,\ 0 \le y \le 1)$ the data $u(x,y) = 2(1-x)$ are used. This yields a contact layer near this boundary. All other boundary conditions were taken homogeneous. This problem is again discretized by the FEM. In figure 5b we show the solution u_h^A and in 5a the solution with artificial diffusion $(\alpha = \varepsilon + h/4)$. We see that by u_h^A the profile both of the interior layer and of the contact layer are well represented.

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