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Multigrid Approaches to the Euler Equations

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In this report we discuss different approaches to solve the Euler equations for compressible flow. The emphasis is on the multigrid acceleration of the solution process for finding approximations to the steady state solution.

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Contents:

1. INTRODUCTION

The efficient solution of flow problems is one of the early aims in the applications of multigrid (MG) methods [4] . However, in recent years most progress in the development of MG has been made in the field of elliptic partial differential equations and other fields where a solid mathematical theory existed (e.g. integral equations). For the inherently more complex equations that describe flow problems, the theoretical development of MG did not proceed at the same pace. Early work was done by Brandt [39, 7, 6] , where e.g. the Stokes equations and the incompressible and compressible Navier-Stokes equations were considered.

On the other hand, triggered by practical interest from the engineering sciences, several attempts have been made to apply MG ideas to improve the efficiency of flow computations. Assuming that the flow is irrotational, it can be described by the potential equation, which -in the interesting case of transonic flow- is of mixed hyperbolic and elliptic type. By the use of MG, substantial improvements were made in the solution procedures for these equations [39, 3, 15, 31, 8, 34] . Dropping the assumption of irrotational flow, an exact description of inviscid flow is given by the Euler equations. When the physical effects of viscosity and heat conduction are also included, these equations extend to the Navier-Stokes equations. Models of turbulence can also be included in the Navier-Stokes equations.

In this chapter we will treat several multiple grid approaches that are used for the solution of the equations of compressible flow. We restrict ourselves to problems in 2 space dimensions. Almost all techniques discussed here can be applied in 3-D as well, but the burden of 3-D notation makes the description unattractive. Also, in practice, most codes are written for 2-D problems because the

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complexity and computational requirements for 3-D computations are at the limit of present day computer capabilities. With the advent of the more powerful computers this situation will certainly change in the near future.

Practical problems that arise in the aircraft and turbomachinery industries are often described by the compressible Navier-Stokes equations. Yet we shall mainly consider the Euler equations of inviscid flow. The reason for this is the assumption that a good method for the solution of the Euler equations may be extended to those situations where viscosity plays a significant role.

In those cases where the solution of the Euler equations can be used as a first approximation to the solution of the full Navier-Stokes equations, it may be a convenient approach to compute (an approximation) to this Euler flow first. Then this approximation can be corrected for viscous effects. Most simply this is done by a defect correction approach [2] , where the solution of the Navier-Stokes equations is found by an iterative process, in which only Euler-type equations are (approximately) solved and the heat conduction and viscous Navier-Stokes terms are taken care of by adding the corresponding corrections as forcing terms. In practice, such procedures are simply realized in an iterative process for the solution of the Navier-Stokes equations by neglecting the extra Navier-Stokes terms at particular stages of the solution process.

The equations

The 2-D *Navier-Stokes equations,* describing the physical laws of conservation of mass, momentum and energy, can be written in conservation form as

$$\frac{\partial}{\partial t} q + \frac{\partial}{\partial x} F(q) + \frac{\partial}{\partial y} G(q) = 0 , \tag{1.1a}$$

where

$$F(q) = f(q) - \mathrm{Re}^{-1}\, r(q) , \qquad G(q) = g(q) - \mathrm{Re}^{-1}\, s(q) , \tag{1.1b}$$

and

$$q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{bmatrix} , \quad f = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uH \end{bmatrix} , \quad g = \begin{bmatrix} \rho v \\ \rho vu \\ \rho v^2 + p \\ \rho vH \end{bmatrix} ,$$

$$r = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \kappa \mathrm{Pr}^{-1}(\gamma-1)^{-1}(c^2)_x + u\tau_{xx} + v\tau_{xy} \end{bmatrix} , \quad s = \begin{bmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ \kappa \mathrm{Pr}^{-1}(\gamma-1)^{-1}(c^2)_y + u\tau_{xy} + v\tau_{yy} \end{bmatrix} .$$

Here ρ , u , v , e and p respectively represent density, velocity in x- and y- direction, specific energy and pressure; $H = e + p / \rho$ is the specific enthalpy. The pressure is obtained from the equation of state, which - for a perfect gas - reads

$$p = (\gamma-1)\rho(e - \tfrac{1}{2}(u^2+v^2)) ;$$

γ is the ratio of specific heats. $q(t,x,y)$ describes the state of the gas as a function of time and space and f and g are the convective fluxes in the x- and y- direction respectively. Re and Pr denote the Reynolds and Prandtl number; thermal conductivity is given by κ; $c = \sqrt{\gamma p / \rho}$ is the local speed of sound; and

$$\tau_{xx} = (\lambda + 2\mu) u_x + \lambda v_y ,$$

$$\tau_{xy} = \mu(u_x + v_x),$$
$$\tau_{yy} = (\lambda + 2\mu)v_y + \lambda u_x,$$

where λ and μ are viscosity coefficients. Stokes assumption of zero bulk viscosity may reduce the number of coefficients by one: $3\lambda + 2\mu = 0$.

We denote the open domain of definition of (1.1) by Ω^* .

The *Euler equations* are obtained from (1.1a) by neglecting viscous and heat conduction effects:

$$F(q) = f(q)\,, \qquad G(q) = g(q)\,. \tag{1.1c}$$

The time dependent Euler equations form a hyperbolic system : written in the quasi-linear form

$$\frac{\partial q}{\partial t} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial g}{\partial q}\cdot\frac{\partial q}{\partial y} = 0\,,$$

the matrix

$$k_1 A + k_2 B = k_1 \frac{\partial f}{\partial q} + k_2 \frac{\partial g}{\partial q} \tag{1.2}$$

has real eigenvalues for all directions (k_1,k_2).

These eigenvalues are $(k_1u+k_2v)\pm c$ and (k_1u+k_2v) (a double eigenvalue). The sign of the eigenvalues determines the direction in which the information about the solution is carried along the line (k_1,k_2) as time develops (i.e. it determines the direction of flow of characteristic information). It locates the direction of the domain of dependence.

It is well known that -because of the nonlinearity- solutions of the Euler equations may develop discontinuities, even if the initial flow $(t=t_0)$ is smooth. To allow discontinuous solutions, (1.1) is rewritten in its integral form

$$\frac{\partial}{\partial t}\iint_{\Omega} q\; dx\; dy \;+\; \int_{\partial\Omega}(\,f.n_x + g.n_y\,)\; ds \;=\; 0\,, \qquad \text{for all } \Omega\subset\Omega^*\;; \tag{1.3}$$

$\partial\Omega$ is the boundary of Ω and (n_x,n_y) is the outward normal vector at the wall $\partial\Omega$.

The form (1.3) of equation (1.1) shows clearly the character of the system of conservation laws: the increase of q in Ω can be caused only by the inflow of q over $\partial\Omega$. In symbolic form we write (1.3) as

$$q_t + N(q) = 0. \tag{1.4}$$

The solution of the weak form (1.3) of (1.1a,c) is known to be non-unique and a physically realistic solution (which is the limit of a flow with vanishing viscosity) is known to satisfy the additional entropy condition (cf. [27,28]). The entropy condition implies that characteristics do not emerge at a discontinuity in the flow.

The *steady state* equations are obtained by the assumption $\partial q\,/\,\partial t = 0$. Guided by the defect correction principle and knowing how the viscous effects change the governing equations, for the Navier-Stokes equations with large Reynolds number, we can concentrate on the solution methods for the stationary Euler equations:

$$N(q) = 0. \tag{1.5}$$

The discretizations

For the discretization of (1.1) or (1.3), two different approaches can be taken. First, the time and space discretization can be made at once. This leads, for example, to discretization schemes of Lax-Wendroff type. An initial state of the fluid, $q^h_{(n)}$, defined on a discrete grid, is advanced over one time-step. Using a second order approximation in time, this yields

$$q^h_{(n+1)} = q^h_{(n)} + \Delta t\,(q^h)_t + \frac{1}{2}(\Delta t)^2\,(q^h)_{tt}\,. \tag{1.6}$$

With the equation (1.1a,c), we arrive at

$$q_{ij}^{(n+1)} = q_{ij}^{(n)} - \Delta t\,(f_x + g_y)_{ij} + \frac{1}{2}(\Delta t)^2 \left\{ [A\,(f_x + g_y)]_x + [B\,(f_x + g_y)]_y \right\}_{ij} ,$$

where A and B are defined by (1.2). Using various difference approximations of the bracketed terms in the right-hand-side, different Lax-Wendroff type discretizations may be obtained.

Typically this type of discretization is made on a rectangular grid. If the domain Ω^* is not rectangular, a 1-1-mapping $(x,y) \longleftrightarrow (\xi,\eta)$ between the physical domain and a rectangular computational domain can be constructed. Then the differential equation and the boundary conditions are reformulated on this computational domain.

A property of most of these Lax-Wendroff discretizations is that, when by time-stepping a stationary state is obtained, such that $q^h_{(n+1)} = q^h_{(n)}$, the discrete stationary state still depends on Δt. This is caused by the fact that the discrete term with $(\Delta t)^2$ in (1.6) in general does not vanish.

A second approach is to distinguish clearly between the time and the space discretization by the method of lines. First, a space discretization is made for the partial differential equation (1.4), by which it is reduced to a large system of ODEs

$$\frac{\partial}{\partial t} q^h = N^h(q^h) . \tag{1.7}$$

Now, to find an approximation of the time-dependent solution of (1.4), any method can be used for the integration of this system of ODEs. The solution of the steady state can be computed by solving (1.7) until the transients have died out. Alternatively, we can avoid the ODEs (1.7) and solve the non-linear system

$$N^h(q^h) = 0 \tag{1.8}$$

by other (more direct) means. In both cases (1.7) and (1.8), we find a steady approximate solution q^h which is independent of the choice of a timestep.

For the construction of the semidiscrete system (1.7) or (1.8) on a non-rectangular domain Ω^*, again a mapping $(x,y) \longleftrightarrow (\xi,\eta)$ can be introduced and finite difference approximations (of an arbitrarily high order) can be used to construct a space discretization of the transformed steady equation

$$[y_\eta F(q) - x_\eta G(q)]_\xi + [-y_\xi F(q) + x_\xi G(q)]_\eta = 0 .$$

Another way to construct the system (1.7) on a non-rectangular grid is by a *finite volume* technique. Here, the starting point for the discretization is (1.3). Without an *a-priori* transformation, the domain Ω^* is divided into a set of disjoint quadrilateral cells Ω_{ij}. The discrete representation q^h of q is given by the values q_{ij}, the (mean) values of q in the cell Ω_{ij}. Using different approximations for the computation of fluxes between the cells Ω_{ij}, various finite volume discretizations are obtained. A conservative scheme is easily obtained by computing a unique approximation for each flux over the boundary between two neighboring cells.

In order to define a proper sequence of discretizations as $h \to 0$ for a non-rectangular grid, a formal relation between the vertices of cells Ω_{ij} and a regular grid can be given, again by a mapping $(x,y) \longleftrightarrow (\xi,\eta)$. If this mapping is smooth enough, it can be proved that for refinements $h \to 0$ which correspond with regular refinements in (ξ,η), space discretizations up to second order can be obtained. An advantage of the finite volume technique is that the untransformed equations can be used, even for a complex region. Boundary condition information is also usually simpler for finite volume methods.

With the finite volume technique, both central difference and upwind type finite volume schemes are used. They differ by the computation of the flux between neighboring cells Ω_{ij}:

(1.) For a central difference type, the flux over a cell wall Γ_{LR} between two cells with states q_L and q_R is computed as $\frac{1}{2} f^*(q_L) + \frac{1}{2} f^*(q_R)$, where $f^* = k_1 f + k_2 g$ is the flux normal to Γ_{LR}. On a

Cartesian grid this scheme reduces to the usual central difference scheme. In order to stabilize this scheme, and to prevent the uncoupling of odd and even cells in the grid, it is necessary to supplement it with some kind of artificial dissipation (artificial viscosity).

(2.) For upwind difference type discretizations, numerical flux functions $f^*(q_L,q_R)$ are introduced to compute the flux over Γ_{LR}. Several functions f^* are possible. They solve approximately the Riemann problem of gas-dynamics: they approximate the flux between two (initially) uniform states q_L and q_R. Approximate Riemann solvers have been proposed by Steger and Warming [42] van Leer [29] , Roe [37] and Osher [35, 36] . A description of these upwind schemes and their properties can be found in the cited literature. For a consistent scheme, $f^*(q,q) = f^*(q)$, i.e. the numerical flux function with equal arguments conforms with the genuine flux function in (1.1c). All these upwind flux-functions have in common that they are pure one-sided if all characteristics point into the same direction, i.e. $f^*(q_L,q_R) = f^*(q_L)$ if the flow of all information is from left to right.

The multiple grid methods

When a multiple grid technique is used to solve the system of nonlinear (differential) equations (1.7) or (1.8), we assume the existence of a nested set of grids. Usually this nesting is such that a set of 2×2 cells in a fine mesh form a single cell in the next coarser one. (No staggered grids!) The coarser grids are used to effect the acceleration of a basic iterative (time marching or relaxation) procedure on the finest grid.

Slightly generalizing the equations (1.7) or (1.8) to

$$\frac{\partial}{\partial t}\, q^h = N^h(q^h) - r^h \tag{1.9}$$

or

$$N^h(q^h) = r^h, \tag{1.10}$$

where r^h denotes a possible correction or forcing term, we can write the basic iterative procedure as

$$q^h_{(n+1)} \leftarrow G^h\,(q^h_{(n)}, r^h). \tag{1.11}$$

The usual coarse grid acceleration algorithm is as follows: starting with an approximation $q^h_{(k)}$ on the finest mesh, and some approximation $q^{2h}_{(0)}$ on the next coarser one (e.g. $q^{2h}_{(0)} = I^{2h}_h q^h_{(k)}$), first an approximate solution is found for the coarse grid problem

$$N^{2h}(q^{2h}) = N^{2h}(q^{2h}_{(0)}) - \hat{I}^{2h}_h(N^h(q^h_{(n)} - r^h)\,, \tag{1.12}$$

and then the value $q^h_{(k)}$ is updated by

$$q^h_{(k+1)} = q^h_{(k)} + I^h_{2h}(q^{2h} - q^{2h}_{(0)})\,. \tag{1.13}$$

The combination of (1.12) and (1.13) is a *coarse grid correction* (CGC) step. The solution q^{2h} of (1.12) can be approximated e.g. by an (accelerated) iteration process on the 2h-grid again.

We shall see in section 2 that, besides this canonical coarse grid acceleration procedure, the coarser grids sometimes play a different role in the acceleration process [33, 22] .

As we saw in the introductory chapter, a multigrid FAS cycle for the solution of (1.10) now consists of the following steps:

(0) start with an approximate solution q^h .

(1) improve q^h by application of ν_d nonlinear (pre-) relaxation iterations to $N^h(q^h) = r^h$.

(2) if the present grid is not the coarsest, improve q^h by application of one coarse-grid-correction step, where the approximation of (1.12) is effected by μ FAS-cycles to this coarser grid problem; if the present grid is the coarsest, simply skip to (3).

(3) improve q^h by application of ν_u nonlinear (post-) relaxation iterations to $N^h(q^h) = r^h$.

The case $\mu=1$ is called a V-cycle; $\mu=2$ yields a W-cycle. A V-cycle with $\nu_d+\nu_u=1$ is called a *sawtooth cycle.*

2. METHODS BASED ON LAX-WENDROFF TYPE TIME STEPPING

A paper by Ni [33] was among the first to apply a MG acceleration to the (isenthalpic) Euler equations. He uses the following time-stepping procedure as a basic iteration. Starting with an initial state $q^h_{(n)}$, where the values $q^{(n)}_{ij}$ are given at the gridpoints, he first computes the following quantities, by means of a control volume centered integration method with fluxes interpolated from corner values:

$$\Delta q_{i+\frac{1}{2},j+\frac{1}{2}} = -\frac{1}{2}\frac{\Delta t}{\Delta x}[(F_{i+1,j} - F_{i,j}) + (F_{i+1,j+1} - F_{i,j+1})] - \frac{1}{2}\frac{\Delta t}{\Delta y}[(G_{i,j+1} - G_{i,j}) + (G_{i+1,j} - G_{i+1,j+1})] , \tag{2.1}$$

$$F_{i,j} = F(q^{(n)}_{i,j}) \quad \text{etc..}$$

These increments then are distributed over the meshpoints, using direction-weighted means (cell-increments are distributed over mesh-point values):

$$\Delta q_{ij} = \frac{1}{4}\sum_{l=\pm 1}\sum_{k=\pm 1}\left[I - k\frac{\Delta t}{\Delta x}A_{i+\frac{k}{2},j+\frac{l}{2}} - l\frac{\Delta t}{\Delta y}B_{i+\frac{k}{2},j+\frac{l}{2}}\right]\Delta q_{i+\frac{k}{2},j+\frac{l}{2}} , \tag{2.2}$$

$$q^{(n+1)}_{ij} = q^{(n)}_{ij} + \Delta q_{ij} .$$

By the use of the Jacobian matrices A and B, this distribution formula has a kind of upwind effect, but for transonic or supersonic cases an artificial damping is still necessary.

Symbolically, this time stepping process (2.1)-(2.2) is described as:

$$\text{compute} \quad \Delta q^h_{\text{cell}} , \tag{2.3a}$$

with cell values $\Delta q_{i+1/2,j+1/2} \approx -\Delta t \int_{\partial\Omega_{i+1/2,j+1/2}} (f{\cdot}n_x + g{\cdot}n_y)\, ds \;/(\Delta x{\cdot}\Delta y)$;

$$q^h_{(n+1)} := q^h_{(n)} + D^h\,\Delta q^h_{\text{cell}} . \tag{2.3b}$$

The operator D^h is the distribution operator that transfers the cell centered corrections to the grid points by means of (2.2).

The coarse grid acceleration as introduced in [33] by Ni deviates from the canonical coarse grid scheme (1.12) (1.13). In [33] the coarse grid correction is obtained by first computing corrections at coarser cells, $\Delta q^{2h}_{\text{cell}}$. This can be done by restriction of Δq^h to the 2h-grid. Then the corrections $\Delta q^{2h}_{\text{cell}}$ are distributed to the coarser meshpoints similar to (2.2), and the coarse grid correction is interpolated to the fine grid.

Thus, here the coarse grid correction reads

$$\Delta q^{2h}_{\text{cell}} := I^{2h}_h\,\Delta q^h_{\text{cell}} , \tag{2.4a}$$

$$q^h_{(n+1)} := q^h_{(n)} + I^h_{2h}D^{2h}\,\Delta q^{2h}_{\text{cell}} . \tag{2.4b}$$

where I^h_{2h} is a (bi-)linear interpolation operator.

Since the coarse grid corrections are based on fine grid residuals, it is obvious that the possible convergence to a steady state yields a solution of the system (1.8).

In the same way the correction procedure can be repeated on progressively coarser grids. Therefore, in (2.4), $2h$ should be replaced by $2^m h$. We notice that the corrections on the different levels may be made independent of each other. This yields the possibility to compute all coarse grid corrections, $m = 1,...,L$, in parallel and to form the correction

$$q^h_{(n+1)} = q^h_{(n)} + \sum_{m=1}^{L} I^h_{2^m h}\, D^{2^m h}\,\Delta q^{2^m h}_{\text{cell}} .$$

at once [43] . When optimal use of modern multi-processor computers is to be made, it is also possible to perform both computations (2.3a) and (2.4) in parallel [25, 43] .

We see that there are still possibilities to form different variants in the Ni-type multigrid Euler solver. First, any other Lax-Wendroff type time-marching procedure can be used for (2.3a). In [21, 9, 23] Johnson applies the popular MacCormack scheme. Further, in (2.4a) various restrictions, I_h^{2h}, can be used. It transfers the values of the fine grid corrections to a single value for each control volume in the coarser grid. Injection of the correction in the main point of the corresponding cell is often used [22] , but also weighted averages are an obvious choice.

Heuristically, the elucidation for the accelerating effect of the corrections (2.4) is, that these coarse grid corrections may move disturbances of the steady state over the distance of many mesh cells in one timestep, whereas the accuracy of the final solution is only determined by the finest grid. Apparently, it is also necessary that the Lax-Wendroff schemes used in combination with this coarse grid correction are (by the choice of a suitable Δt or otherwise) sufficiently dissipative to reduce the high frequency disturbances that are present in the initial approximation and those introduced during the process by the linear interpolation in (2.4b). Up to now, no complete mathematical theory has been developed to explain and quantify the amount of acceleration, which is clearly found in the many computations that use the described method.

As an alternative to (2.2), where Jacobians are used to form the correction, Johnson [24] introduced a correction that is based on extrapolation (in time) of the computed fluxes.

3. METHODS BASED ON SEMIDISCRETIZATION AND TIMESTEPPING

When only the solution of the steady state is to be computed, the time-accurate integration of the system of ODEs is wasteful. The convergence of (1.4) to steady state is slow. However, the desire to have a procedure that solves transient as well as steady state problems, coding convenience, or the restrictions imposed by the optimal use of vector computers may be a reason to prefer time-stepping methods. When no time accuracy is desired, many devices are known to accelerate the integration process (cf. [38]). For the solution of the Euler equations, these devices include: (i) local timestepping, which means that the stepsize in the integration process may differ over different parts of the domain Ω^*; (ii) enthalpy damping, where a-priori knowledge about the behavior of the enthalpy over Ω^* is used (e.g. H constant over Ω^*); (iii) residual smoothing, and (iv) implicit residual averaging, where the fact is used that instability effects appear first for high frequencies, so that larger timesteps are possible when the residual is smooth.

For all explicit integration methods, stability requirements set a limit to the size of the possible timesteps (CFL limits). Implicit integration procedures can be unconditionally stable, but they require the solution of a (nonlinear) system in each individual timestep.

An important code, based on a time-stepping method has been developed by Jameson, Schmidt and Turkel [17] They use an explicit time-stepping method of Runge-Kutta type. This *multistage timestepping procedure* is a specially adapted Runge-Kutta method, where the hyperbolic (=convective) and the parabolic (=dissipative) parts of $N^h(q^h)$ are treated separately. The Runge-Kutta coefficients in the k-stage Runge-Kutta schemes (k= 3,4), are selected not only for their large stability bounds, but also with the aim to improve the damping of the high frequency modes. In the k stages of the Runge-Kutta process, the updating of the dissipative part is frozen at the first stage. This saves a substantial part of the computational effort.

The multigrid scheme used by Jameson [16] is a FAS sawtooth cycle with $\nu_d=1$. The restriction I_h^{2h} ($\tilde{I}_h^{2h}$) is defined by volume-weighted averaging of the states (summation of changes of states, respectively). The prolongation I_{2h}^h is defined by bilinear interpolation. The basic smoothing procedure is the "multistage time-stepping scheme". On the coarser grids the stability bounds for the timestep, which are $\mathcal{O}(h)$, allow larger timesteps. On each grid the timestep is varied locally to yield a fixed Courant number, and the same Courant number is used on all grids, so that progressively larger timesteps are used after each transfer to a coarser grid. As for Ni's method, the reasoning is that disturbances from the steady state will be more rapidly expelled from the domain Ω^* by the larger timesteps. The interpolation of corrections back to the fine grid introduces high frequency errors, which cannot be rapidly expelled. These errors should be locally damped. Hence, to obtain a fast rate of convergence, the time-stepping process should rapidly damp the high frequency errors.

In [20] Jespersen announces an interesting theorem on the use of the MG process in combination with a time-stepping procedure. This theorem asserts the following. Let $N^h(q^h) = 0$ be a space discretization of $N(q) = 0$, which is consistent, i.e.

$$N^h(I^h(q)) - \hat{I}^h N(q) = \Theta(h),$$

and let the time-stepping procedure be consistent in time

$$q^h_{(n+1)} = q^h_{(n)} + \Delta t_{(n)} [N^h(q^h_{(n)}) - r^h] + \Theta((\Delta t_{(n)})^2).$$

If we consider the sawtooth algorithm, with $\nu_d = 1$, $\nu_u = 0$, $\mu = 1$, and if I_h and I^h satisfy an approximation property (i.e. for a smooth function q the prolongation and restriction in the state space are such that $I_h I^h q - q = \Theta(h)$), then the MG algorithm on L grids is a consistent, first order in time, discretization of (1.4) with timestep $\Delta t_{tot} = \sum_{j=1,\ldots,L} \Delta t_j$.

This theorem formalizes in a sense the heuristic reasoning that on coarser grids the deviations from steady state can be expelled faster by the use of larger timesteps. This may suggest that more, say $k > 1$, steps on the coarser grids would improve the convergence even more. However, the theorem regards consistency; stability is not considered. Hence, in the same paper [20] Jespersen shows by an example that convergence is lost when a large number of relaxations is made on the coarse grid. In fact a strong stability condition of the form $\Delta t / \Delta x \leqslant \Theta(k^{-1})$ seems to appear.

4. FULLY IMPLICIT METHODS

Most methods so far developed are based on the concept of integrating the equations (1.4) in time until a steady state is reached. If we are only interested in a possible solution of the steady state equation (1.5) and assume that this solution is unique, we may disregard the time-dependence completely. Further, assuming that a suitable space discretization takes into account the proper directions of dependence in Ω^*, we can restrict ourselves simply to the solution of the nonlinear system (1.8) or

$$N^h(q^h) = r^h. \tag{1.10}$$

Also, if the time-dependent system (1.9) is solved by means of an implicit time-stepping method -in order to circumvent the stability bounds on Δt-, we have to solve systems (1.10) at each step time step. As soon as we mix with these implicit solution methods and give up time accuracy for (1.10), there is little or no difference between these time stepping procedures and (nonlinear) relaxation methods for (1.10).

Starting with the nonlinear system (1.10), two direct MG approaches are open. We can either apply the nonlinear multiple grid algorithm (FAS) directly to the system (1.10) or we may apply linearization (Newton's method) and use the linear version of multiple grid (CS) for the solution of the resulting linear systems. Jespersen [19] gives an extensive recital of the (dis)advantages of both approaches. Both have been used with success for the Euler equations.

Linearization and CS have been used by Jespersen [18] and Mulder [32] ; the non-linear FAS procedure is used by Steger [41] , Jespersen [18] and Hemker-Spekreijse [14,13] .

In all these papers upwind discretizations have been used. In [18,41] the Steger-Warming scheme is used; [32] uses the differentiable van Leer flux-splitting method; [14,13] use Osher's flux difference splitting. In [10] Dick also considers Roe's flux difference splitting for the 1D Euler equations.

When Newton's method is applied for linearization, it may be difficult to start in the domain of contraction of the iteration. Therefore, Mulder [32] introduces the so called Switched Evolution

Relaxation (SER) scheme, which is a chimera of a forward Euler time-stepping and a Newton method:

$$\left[\frac{1}{\Delta t} I - \frac{\partial}{\partial q} N^h(q^h_{(n+1)})\right] (q^h_{(n+1)} - q^h_{(n)}) = N^h(q^h_{(n)}). \tag{4.1}$$

For $\Delta t \to 0$, this gives the simple time stepping procedure; for $\Delta t \to \infty$, (4.1) is equivalent to Newton's method. In the actual computation Δt varies, depending on the size of the residual, such that (4.1) is initially a time stepping procedure and becomes Newton's method in the final stages of the solution process.

In a FAS procedure, a natural way to obtain an initial estimate is -of course- the use of Full Multi-Grid (FMG) [5]. The initial estimate is obtained by interpolation from the approximate solution on the coarser grid(s). For many problems this process gives very good results, even if one starts with rough approximations on a really coarse grid.

A nested sequence of Galerkin discretizations.

When (1.3) is discretized by a finite volume method, and if a conservative first order upwind (or a central difference) discretization is used as described in section 1, it can be shown [13] that with a particularly simple restriction $\hat{I}_h^{2h}$ and prolongation $P2hh$, the coarse discrete operator N^{2h} is a Galerkin approximation to the fine grid discretization N^h. With I^h_{2h} the piecewise constant interpolation over cells, and $\hat{I}_h^{2h}$ the addition of the residual over fine mesh cells to form a residual on the corresponding coarse cell, the following relation holds

$$N^{2h}(q^{2h}) = \hat{I}_h^{2h} N^h(I^h_{2h} q^{2h}). \tag{4.2}$$

This formula has an interesting implication for a coarse grid correction that is constructed by means of these operators. If the coarse grid correction (1.12)-(1.13) transforms the approximation q_h into $\tilde{q}^h$, the residual of $\tilde{q}^h$ satisfies

$$\hat{I}_h^{2h}\left[r^h - N^h(\tilde{q}^h)\right] \tag{4.3}$$

$$= \hat{I}_h^{2h}\left[\left(N^h q^h - N^h I^h_{2h} I^{2h}_h q^h\right) - \left(N^h \tilde{q}^h - N^h I^h_{2h} I^{2h}_h \tilde{q}^h\right)\right].$$

For a smooth operator N^h, this implies

$$\hat{I}_h^{2h} [r^h - N^h(\tilde{q}^h)] = \mathcal{O}(\| q^h - \tilde{q}^h \|^2).$$

This means that the restriction of the residual mainly contains high frequency components. As is the case with common elliptic problems, it is the task of the relaxation method to efficiently damp these highly oscillating residuals.

Relaxation methods

Clearly, whether a sequence of Galerkin approximations is used or not, the important feature for a relaxation method in a multiple grid context (both CS and FAS) is its capability to damp the high frequency components in the error (or in the residual). Therefore the difference scheme should be sufficiently dissipative. The first order upwind schemes usually are. An advantage of these schemes over central differences is that this numerical dissipation is well defined and independent of an artificial parameter for the added dissipation, which is necessary for the central difference schemes. The lack of differentiability of the numerical flux function may create a problem, but some differentiable flux functions are now available [41,35,36,29].

Both in the linearized (CS) and in the nonlinear (FAS) application, well-known and simple relaxation procedures such as Gauss-Seidel (GS), symmetric Gauss-Seidel (SGS) and line-Gauss-Seidel (LGS) are reported to work well when applied to the discrete Euler equations. (All these relaxation methods are used in their "collective" version, i.e. the 3 or 4 variables corresponding to a single point (or cell) are relaxed simultaneously.) The smoothing behavior of these relaxations can be analyzed by Local Mode Analysis. Here we should notice that the *smoothing factor* as used for common elliptic problems, has no significant meaning for the Euler equation because we have to take into account characteristic (unstable) modes. A local mode analysis should follow more the lines used for singularly perturbed elliptic problems, cf. e.g. [26] . Jespersen [18] has published some results. He shows that for a subsonic and a supersonic case SGS has a reasonably good smoothing behavior, when applied to a first order scheme. Of course, the non-symmetric GS relaxation is only effective if the direction of the relaxation sufficiently conforms with the direction of the characteristics. If we study plots of reduction factors of Fourier components (spectral radii, or norms for the error or residual amplification operator), e.g. when SGS is applied to the Euler equations, we see that two SGS sweeps are usually sufficient for a significant reduction of the high frequencies (Hemker, unpublished results). For second order schemes the smoothing rates are not satisfactory.

Mulder and van Leer published a study [30] where several relaxation schemes (GS, LGS, ZEBRA, point-Jacobi, line-Jacobi, ADI, AF) were compared, when applied to the linearized isenthalpic Euler equations.

Higher order schemes

When both first and second order upwind schemes are studied, the best MG performance is found for the first order discretizations. This can be explained by the fact that first order upwind schemes are more dissipative and hence more able to damp high frequencies. As first order schemes may not be accurate enough for practical computations and, moreover, have the unpleasant property to smear out skew discontinuities, second order schemes are highly desired.

Beside the possibility of applying the MG acceleration directly to the second order scheme -with the unwanted effect of the slow down of the convergence rate- another possibility exists. Starting with a first order approximation, we can improve the accuracy by defect correction iteration [2, 12, 40]

$$N_1^h(q_{(n+1)}^h) = N_1^h(q_{(n)}^h) - N_2^h(q_{(n)}^h). \tag{4.4}$$

Here N_p^h, p = 1,2, denotes the p-th order discretization. A theorem exists [11] , which shows that for smooth solutions a single correction step (4.4) is sufficient to obtain the higher order of accuracy. Also for solutions with discontinuities (where the formal order of convergence has no practical meaning) it is shown in [12] that one or a few steps (4.4) improve the accuracy of the solution significantly.

In the following table we summarize the several attempts to solve the steady Euler equations by an MG method with implicit relaxation. It is our opinion that the recent methods of this class are the most robust and efficient ones to solve the steady Euler equations. The development in the last few years has led to a significant improvement of the algorithms. However, the fully implicit methods have a rather complex structure and are not directly suited for vector computers. Furthermore, at the moment the practical experience with these methods is much less than e.g. with Jamesons multistage timestepping procedure or the commonly used Beam-Warming [1] algorithm.

	Discretization scheme	MG	Relaxation
Steger (1981)	Steger-Warming Finite differences	FAS	AF
Jespersen (1983)	Steger-Warming Finite differences	FAS/ CS	SGS, GS
Mulder (1984)	van Leer Finite differences	CS	SGS
Hemker & Spekreijse (1985)	Osher Finite volumes	FAS nested Galerkin	SGS, Damped Jacobi

Table 1. Fully implicit Multiple Grid approaches

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