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Line Gauss-Seidel Relaxation and Multigrid for Steady, Two-Dimensional Flow Computations

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Convergence results are presented for steady, 2D Euler and Navier-Stokes flow computations with line Gauss-Seidel relaxation accelerated by multigrid. Comparisons are made with point Gauss-Seidel relaxation. For Euler flow computations, line relaxation appears to lead to a more efficient multigrid technique than point relaxation. For Navier-Stokes flow computations, the advantage of line relaxation is its greater robustness.

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1. Introduction

1.1. Governing equations

The flow equations considered are:

$$\frac{\partial}{\partial x}\begin{pmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho u(e+\frac{p}{\rho})\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ \rho v(e+\frac{p}{\rho})\end{pmatrix}- \tag{1a}$$

$$\frac{1}{Re}\left[\frac{\partial}{\partial x}\begin{pmatrix}0\\ \tau_{xx}\\ \tau_{xy}\\ \tau_{xx}u+\tau_{xy}v+\frac{1}{\gamma-1}\frac{1}{Pr}\frac{\partial(c^2)}{\partial x}\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}0\\ \tau_{xy}\\ \tau_{yy}\\ \tau_{yy}v+\tau_{xy}u+\frac{1}{\gamma-1}\frac{1}{Pr}\frac{\partial(c^2)}{\partial y}\end{pmatrix}\right]=0,$$

with

$$\tau_{xx}=\frac{4}{3}\frac{\partial u}{\partial x}-\frac{2}{3}\frac{\partial v}{\partial y}, \tag{1b}$$

$$\tau_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},$$

$$\tau_{yy}=\frac{4}{3}\frac{\partial v}{\partial y}-\frac{2}{3}\frac{\partial u}{\partial x}.$$

For a detailed description of the various quantities used, assumptions made and so on, we refer to any standard textbook. Suffice to say that these are the full, steady, 2D, compressible Navier-Stokes equations with zero bulk viscosity and constant diffusion coefficients. For $1/Re=0$, diffusion has vanished and the remaining equations are the Euler equations.

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1.2. Discretization method
For both the Euler and Navier-Stokes equations, we have considered first- and higher-order accurate discretizations [3,5,7,8]. Since for both the first- and higher-order discretized equations, the relaxations are performed on the first-order discretized equations only, here we limit the discussion only to that case. The discussion is concise, for details we refer to [5,7].

Both the Euler and Navier-Stokes equations are discretized in their integral form. The discrete system of equations is obtained by subdividing the integration region into finite volumes, and by requiring that the integral form holds for each finite volume separately. The resulting discrete operator implies the evaluation at each finite volume wall of the convective flux vector and, additionally for Navier-Stokes, the diffusive flux vector.

For the evaluation of the convective flux vector we use an upwind approach, following the Godunov principle [1]. For the solution of the resulting 1D Riemann problem, Osher's approximate Riemann solver [9] is preferred. First-order accuracy for the convection part is considered only. It is simply obtained by taking the left and right state at each finite-volume wall equal to the one in the corresponding adjacent volume.

For the evaluation of the diffusive flux vector, we use the central finite-volume technique as outlined in [10]. This technique is second-order accurate.

1.3. Earlier developed solution method
The solution method used so far for the first-order discretized equations, is a multigrid method with collective point Gauss-Seidel relaxation as the smoothing technique. Briefly summarized, the multigrid method applied is nonlinear multigrid iteration (FAS) preceded by nested iteration (FMG). Its cycles are of V-type and have a single pre- and post-relaxation per grid level. For details we refer to [5,6]. In the point Gauss-Seidel relaxation, per finite volume, one or more Newton iteration steps are performed for the collective relaxation of the four state-vector components. (Usually, the tolerance for the Newton iteration is so large that in a substantial majority of all cells, only a single Newton step is performed.) In general, for the first-order discretized equations, collective point Gauss-Seidel relaxation appears to be a good smoother. Therefore, in many cases, it allows a good acceleration by multigrid. However, recently we experienced the following (already expected) deficiencies of the point relaxation method:

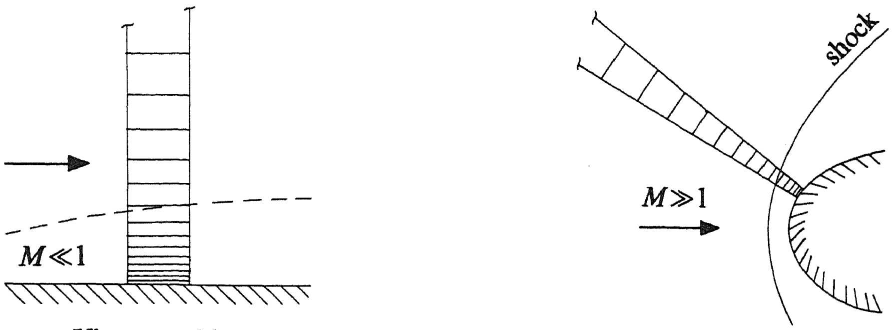
- In very low-subsonic flow regions (regions in which $\sqrt{u^2+v^2}/c \ll 1$, as for instance stagnation regions and viscous sublayers adjacent to the wall) the derivative matrix used in the Newton iteration is ill-conditioned and becomes singular for $\sqrt{u^2+v^2}/c \to 0$. A clear illustration of this is given in [4].
- In the initial phase of a steady flow computation in which strong perturbations of the solution arise, a local iterand may be easily swept out of Newton's convergence range and, thus, may cause global divergence. (This may easily happen for instance in the computation of a hypersonic blunt body flow which has been - crudely - initialized to its hypersonic upstream flow conditions and in which a strong shock wave is arising.)

2. New solution method

2.1. Line relaxation
If aforementioned situations are really of a local nature, line relaxation may be a robust remedy. In a local, very low-subsonic flow region such as a viscous sublayer adjacent to the wall, lines crossing that layer and running into the outer solution (Fig. 1a) are affected to a smaller extent by the low speeds than single volumes in that layer. For a strong hypersonic shock wave arising in an initially unperturbed flow field, a similar reasoning may hold for lines crossing the shock wave and running to the far-field boundary (Fig. 1b), and single volumes in, or downstream, of that shock wave.

In a viscous sublayer with high aspect ratio volumes (such as in Fig. 1a), an additional advantage of the properly directed line relaxation mentioned is that it is well-adapted to the corresponding strong coupling in crossflow direction. In convection dominated flow regions, a strong coupling exists in flow direction. Here, lines are to be preferred which are more or less aligned with the flow. So, if well-aligned, this is an additional advantage of lines crossing shock waves.

With line Gauss-Seidel relaxation as a new smoother, the earlier developed multigrid technique can be maintained.



a. Viscous sublayer b. Hypersonic shock wave

Fig. 1. Relaxation lines crossing difficult flow region

2.2. Relaxation matrix

For line relaxation applied in an Euler flow computation, two basic types of flow can be distinguished: flows with either subsonic or supersonic velocity components along the line considered. For the subsonic case, Osher's scheme (correctly) picks up information from up- and downstream direction, with as a consequence a block-tri-diagonal relaxation matrix. For the supersonic case the result is a block-bi-diagonal matrix. For Navier-Stokes flow computations a block-tri-diagonal matrix is the result in any case, except in the rare case of supersonic velocity components and zero gradients of u,v and c^2 along the line. (Then a block-bi-diagonal matrix results again.) In all cases the blocks are 4×4-matrices.

No special effort was put into an efficient implementation of the solution method for the block-diagonal system; a solver for a general band-matrix is applied.

3. CONVERGENCE RESULTS

As test case to study the convergence of the method we consider a supersonic flat plate flow with an oblique shock wave impinging upon the plate (Euler) or boundary layer (Navier-Stokes). The test case stems from [2]. The particular experiment considered in [2] is that at $M=2$, $Re=2.96\ 10^5$. Since in this experiment, the flow is known to be laminar but yet hard to compute (because of the shock induced separation), it is a benchmark problem for laminar, 2D, compressible Navier-Stokes codes. The grids used for Euler and Navier-Stokes are identical. The coarsest grid applied in all multigrid computations is the 5×2-grid (Ω_0) shown in Fig. 2a. A fine grid considered is the $2^4(5\times2)$-grid (Ω_4) shown in Fig. 2b.

a. Ω_0

b. Ω_4

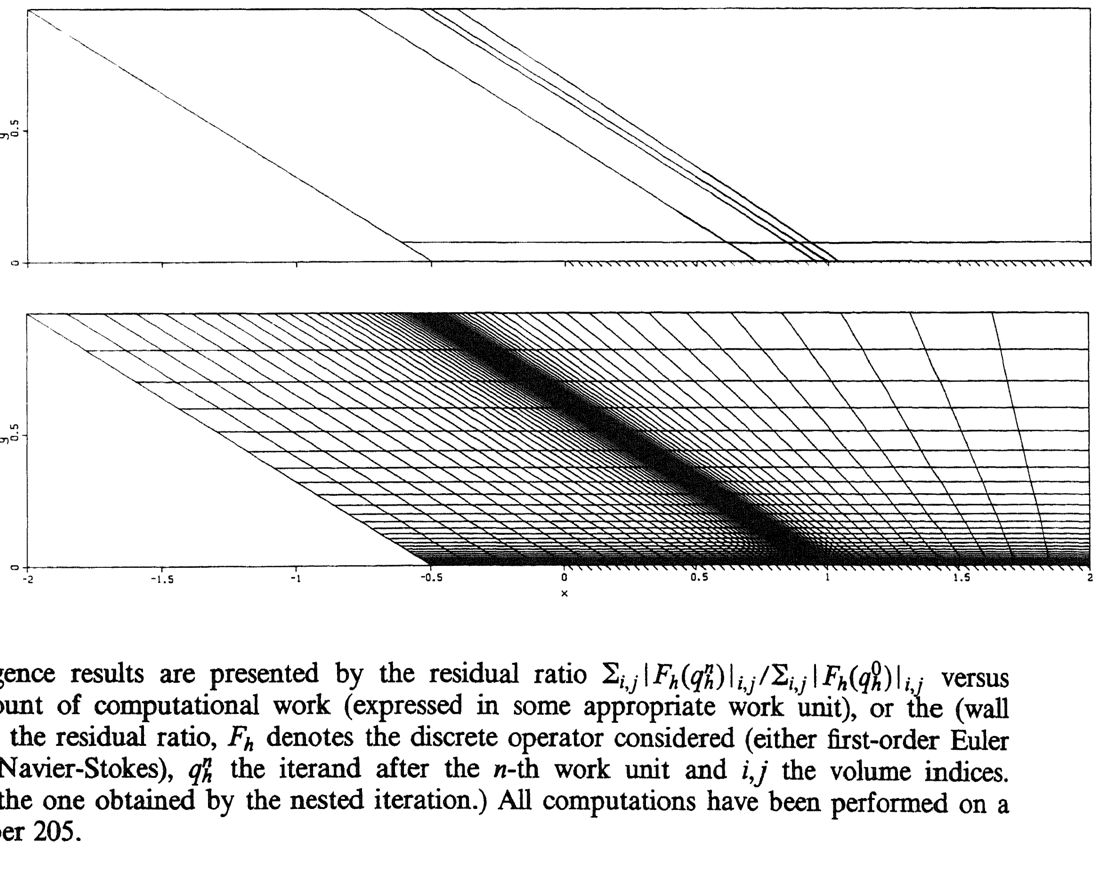


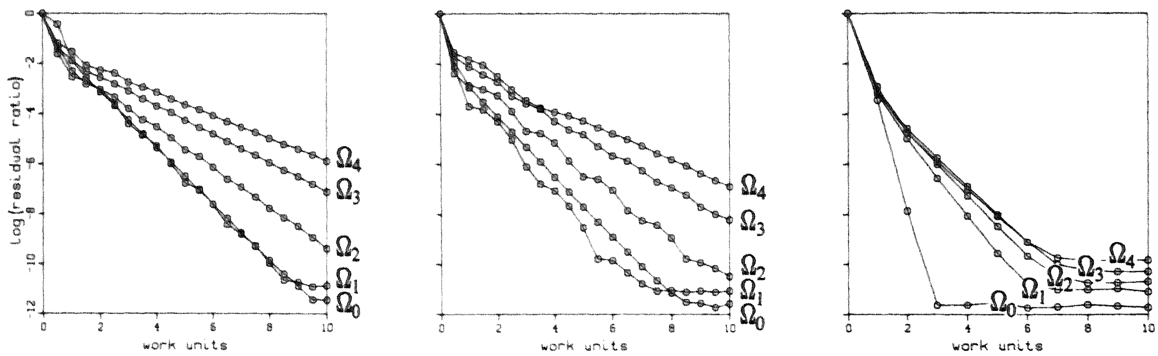
Fig. 2. Grids considered

The convergence results are presented by the residual ratio $\Sigma_{i,j}|F_h(q_h^n)|_{i,j}/\Sigma_{i,j}|F_h(q_h^0)|_{i,j}$ versus either the amount of computational work (expressed in some appropriate work unit), or the (wall clock) time. In the residual ratio, F_h denotes the discrete operator considered (either first-order Euler or first-order Navier-Stokes), q_h^n the iterand after the n-th work unit and i,j the volume indices. (Iterand q_h^0 is the one obtained by the nested iteration.) All computations have been performed on a (two-pipe) Cyber 205.

3.1. Euler flow

For the Euler flow, the multigrid behaviour for Gauss-Seidel relaxation with successively points, crosswise lines and streamwise lines, is given in Fig. 3. The streamwise line relaxation is symmetric whereas the other two relaxations are asymmetric with natural downwind sweeps only. To ensure a good comparison of convergence rates, we define a work unit to be equal to: a single multigrid cycle with symmetric relaxation, and consequently: two multigrid cycles with downwind relaxation only.

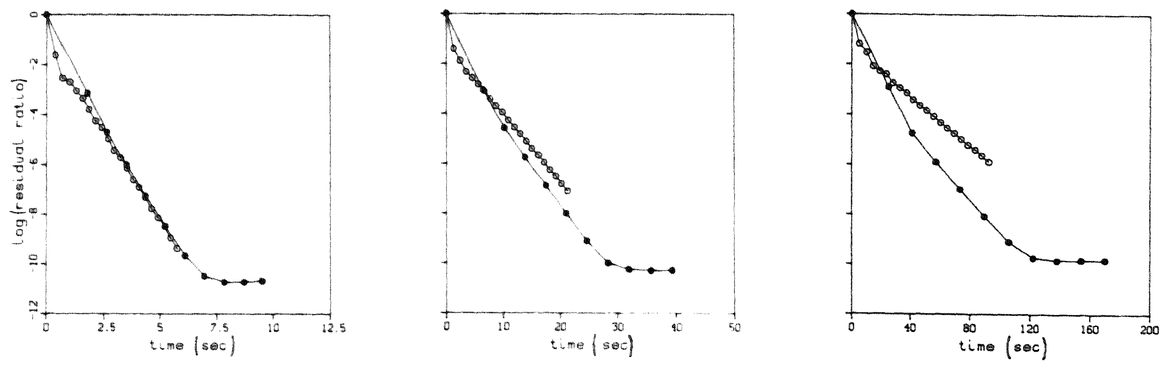
Clearly visible in Fig. 3 are the expected superior convergence rates of the streamwise line relaxation. Also remarkable is its better grid independence.



a. Points b. Crosswise lines c. Streamwise lines

Fig. 3. Multigrid behaviour for three types of Gauss-Seidel relaxation (Euler flow)

In Fig. 4, for three different grids, the efficiency of the streamwise line relaxation is compared with that of the point relaxation. The markers correspond to those in Fig. 3. Though no special effort was put into an efficient implementation of the line relaxation, its efficiency is the same for Ω_2 and better for Ω_3 and Ω_4. (The gain in efficiency on finer grids is of course a consequence of the better grid independence.)



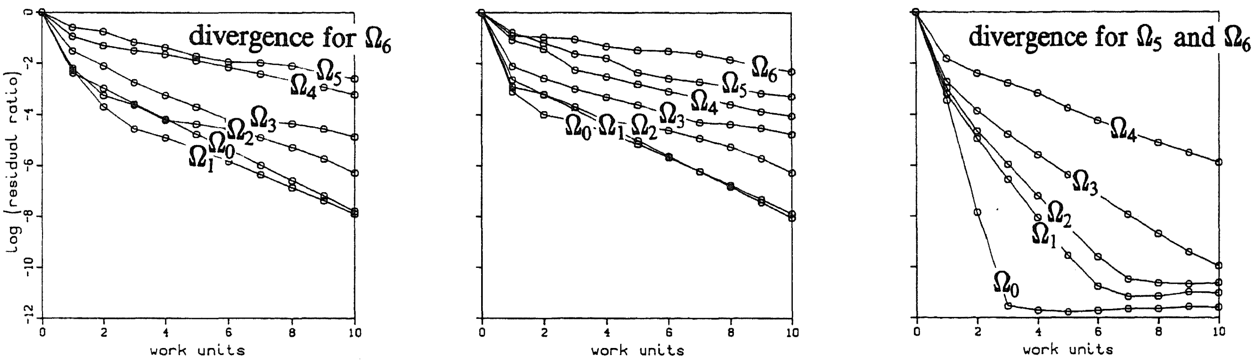
a. Ω_2 b. Ω_3 c. Ω_4

Fig. 4. Convergence histories for point (○) and streamwise line (●) Gauss-Seidel relaxation (Euler flow)

3.2. Navier-Stokes flow

Also for the Navier-Stokes flow, we consider the multigrid behaviour for point, crosswise line and streamwise line relaxation. Here, all relaxations are symmetric, because of the possibly arising subsonic sublayer. Further, here the finest grid considered is Ω_6. A work unit is defined as one multigrid cycle with symmetric relaxation. The convergence rates are given in Fig. 5.

Both for point Gauss-Seidel and streamwise line Gauss-Seidel we have divergence at Ω_6. For the latter relaxation we find also divergence for Ω_5. In both cases the cause is the increasing ill-conditioning directly above the plate with decreasing mesh size normal to the plate. Here the crosswise line relaxation turns out to be more robust.



a. Points

b. Crosswise lines

c. Streamwise lines

Fig. 5. Multigrid behaviour for three types of Gauss-Seidel relaxation (Navier-Stokes flow)

An objection that can be made against the use of crosswise line relaxation throughout the computation, is that though it is well-adapted to the strong coupling in the viscous sublayer with its high aspect ratio volumes, it is not well-adapted to the opposite coupling in the outer flow. (There, streamwise lines are preferred.) The switch in direction of strong coupling suggests an adaptive local line relaxation to be optimal (Fig. 6).

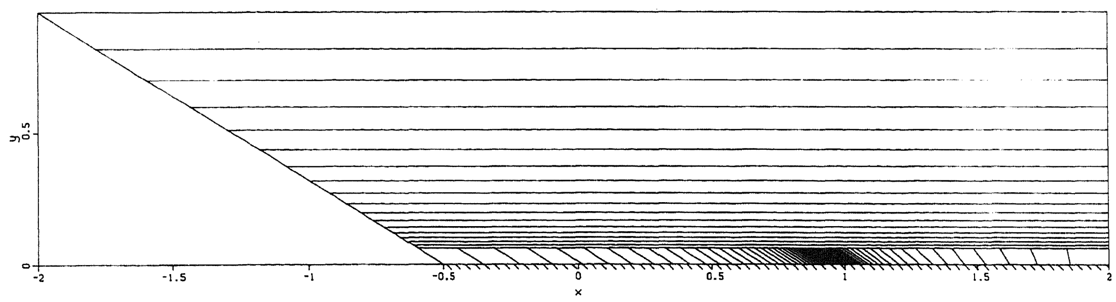


Fig. 6. Locally adapted lines

4. Conclusions

For Euler flow computations, line relaxation appears to be a better smoother than point relaxation if the lines considered are well-aligned with the flow. Already with a relatively slow solver for the large linear system, line relaxation may be more efficient than point relaxation.

For Navier-Stokes flow computations with a practically relevant resolution of viscous layers, the advantage of proper (i.e. crosswise) line relaxation clearly is its greater robustness. It is less sensitive to a strong local ill-conditioning of the flow equations. This smaller sensitivity (greater robustness) probably holds in general for strong local perturbations arising in an initially unperturbed flow.

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