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Abstract

In this paper we analyse the approximation of functions on partially ordered sequences of regular grids. We start with the formulation of minimal requirements for useful grid transfer operators in such a partially ordered context, and we continue with the introduction of hierarchical decompositions and the identification of piecewise constant and piecewise linear approximations as special instances of the tensor product case.

In the second part of the paper we derive error estimates for approximation in different norms on more-dimensional dyadic sequences of regular and sparse grids. We give special attention to a convenient notation.

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Note: The major part of this report will be published elsewhere.

1. Introduction and notation

1.1. Introduction

Recently, in the research on multigrid methods for problems in three dimensions more and more attention is paid to semi-coarsening [5, 13, 15, 19] and sparse grid approaches [2, 6, 8, 9, 11, 18, 14]. This can be understood if we notice that the classical multigrid approach, where a linear sequence of nested grids is used for the approximation on different grids, requires very strong relaxation techniques. The selection of a suitable relaxation is difficult because of the large number of possible choices, each with their particular advantages and disadvantages.

In semi-coarsening, where different coarser grids are introduced, each coarsened in a single direction, the role of the smoothing procedure is reduced, and simpler relaxation procedures can be applied [12]. This makes it attractive to study partially ordered sets of grids, rather than sequentially ordered ones.

Another difficulty, that particularly arises when regular grids are used for the approximation of functions, is the *curse of three dimensions*: the number of cells increases cubically with each refinement in all directions. This results in enormous amounts of degrees of freedom in the approximation, and in very large systems of algebraic equations to solve. This difficulty can be removed to a large extent by adaptive refinement, i.e. by adding only those degrees of freedom that contribute significantly to the improvement of the accuracy. Of course, what degree of freedom adds to the higher accuracy depends on the choice of the set of basis-functions that span the approximating function space. If, on a regular rectangular grid, a hierarchical basis is chosen, for a sufficiently smooth function the degrees of freedom associated with a "sparse grid" are the optimal choice. The sparse grid can be seen as a combination of regular grids, each with a different cell aspect ratio.

In this way semi-coarsening multigrid and sparse grid approximations are much related and make an interesting match.

It appears that the relations between the approximations on the different grids in the partially ordered set are not always clear [4], and that the requirements for the prolongations and restrictions between the approximations on the different grids are often chosen in an *ad hoc* way. In this paper we study the approximation of functions on partially ordered sets of regular grids (on a *grids of grids*). In particular we are interested in the minimal requirements that are needed to introduce the necessary grid transfer operators. Analysing these requirements results naturally in the introduction of a hierarchical decomposition of the approximation on the grid of grids, and we are able to show how the usual approximation by piecewise constant and piecewise linear basis-functions appear as a special case of tensor product form. At that point we can also introduce an MRA (multi-resolution analysis) in more dimensions, for a partially ordered set of approximations.

In the next sections we concentrate on the piecewise constant and piecewise linear approximation. We define their construction in a systematic way and we derive error estimates for the approximations in different norms.

Studying the approximation on a grids of grids, it appeared that a simple and convenient notation was lacking and that the data structures that are used in practice to realise the related algorithms, are rather complicated. Therefore, in the treatment much attention is given to a convenient notation, that can be used in general for the description and analysis of algorithms on a grid of grids.

Sparse grids yield a way for obtaining an approximation with a high accuracy relative to the number of degrees of freedom (unknowns) used. This was first observed by Smoljak [16] for numerical integration and interpolation with trigonometric functions. A different approach of constructing sparse grids is presented in [18]. This approach uses hierarchical basis functions for interpolation with piecewise multilinear functions. Error estimates in different norms and with different assumptions are found in [2, 14, 18]. For obtaining optimal estimates, it is necessary to assume that suitable derivatives of the functions are bounded. In case of singularities these assumptions may not hold. Then, optimal estimates can be obtained on adaptive sparse grids, which can be constructed in a natural way with hierarchical basis functions [2]. High order finite elements on sparse grids were analysed for example in [17].

1.2. Notation

Let $\boldsymbol{k}\in\mathbb{Z}^d$ be a *multi-integer* in d dimensions, then $\boldsymbol{k} = (k_1, k_2, \cdots, k_d)$, with $k_i\in\mathbb{Z}$ for $i = 1, 2, \cdots, d$. We define *relational operators* between multi-integers by

$$\boldsymbol{k} < \boldsymbol{m} \Leftrightarrow (k_1 < m_1 \text{ and } k_2 < m_2 \text{ and } \cdots \text{ and } k_d < m_d)\,,$$

and analogously we define $\boldsymbol{k} \le \boldsymbol{m}$, $\boldsymbol{k} > \boldsymbol{m}$, $\boldsymbol{k} \ge \boldsymbol{m}$ and $\boldsymbol{k} = \boldsymbol{m}$. Further we define

$$\max(\boldsymbol{m}, \boldsymbol{n}) = (\max(m_1, n_1), (\max(m_2, n_2), \cdots, (\max(m_d, n_d))\,,$$

and $\min(\boldsymbol{m}, \boldsymbol{n})$ similarly. In a few instances we will use these operators with the same meaning for real vectors $\boldsymbol{x} = (x_1, \cdots, x_d)\in\mathbb{R}^d$.

With $\boldsymbol{n} = (n_1, \cdots, n_d)\in\mathbb{Z}^d$ we denote $|\boldsymbol{n}| = n_1+\cdots+n_d$. We also use the notation $\mathbf{o} = (0, \cdots, 0)\in\mathbb{N}^d$; $2^{\boldsymbol{n}} = (2^{n_1}, \cdots, 2^{n_d})$; $2^{\boldsymbol{n}}\boldsymbol{x} = (2^{n_1}x_1, \cdots, 2^{n_d}x_d)$; $\boldsymbol{n} \circ \boldsymbol{m} = \sum_{i=1,\ldots,d} n_i m_i$, and $\|\boldsymbol{n}\| = n_1 \cdots n_d$. Further we introduce in $\mathbb{Z}^d$ the unit vectors $\boldsymbol{e}_k$, $k = 1, \cdots, d$, as follows: $\boldsymbol{e}_1 = (1, 0, \cdots, 0)$; $\boldsymbol{e}_2 = (0, 1, 0, \cdots, 0)$; $\boldsymbol{e}_d = (0, \cdots, 0, 1)$, and we use $\boldsymbol{e} = (1, \cdots, 1)$. Finally we define $\mathbf{E} = \{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_d\}$.

Let either $\Omega = \mathbb{R}^d$ be the d-dimensional Euclidean space, or let $\Omega = (0, 1)^d \subset \mathbb{R}^d$ be the d-dimensional open unit cube. With any multi-integer $\boldsymbol{n}\in\mathbb{Z}^d$ we associate a function space $V_{\boldsymbol{n}}$, e.g. the space of piecewise constant or piecewise linear (bi-linear, tri-linear, d-linear) functions on a uniform grid with mesh size $\boldsymbol{h} = (h_1, \cdots, h_d) = (2^{-n_1}, \cdots, 2^{-n_d})$. These grids are uniformly spaced in each of the d coordinate directions, but possibly with a different mesh size in the different directions. The volume of these cells is denoted by $\|\boldsymbol{h}\| = 2^{-|\boldsymbol{n}|}$. The functions in $V_{\boldsymbol{n}}$ all are constant or d-linear on each *dyadic block* or *cell*

$$\Omega_{\boldsymbol{n},\boldsymbol{k}} = [k_1 2^{-n_1}, (k_1 + 1)2^{-n_1}] \times \cdots \times [k_d 2^{-n_d}, (k_d + 1)2^{-n_d}]\,,$$

and this family of cells forms the *grid* $\Omega_{\boldsymbol{n}} = \{\Omega_{\boldsymbol{n},\boldsymbol{k}} \mid \Omega_{\boldsymbol{n},\boldsymbol{k}} \subset \Omega,\ \boldsymbol{k}{\in}\mathbb{Z}^d\}$. The family of *cell centers* or *cell nodes* is denoted by $\Omega^*_{\boldsymbol{n}} = \{\boldsymbol{z}_{\boldsymbol{n},\boldsymbol{k}} \mid \boldsymbol{z}_{\boldsymbol{n},\boldsymbol{k}} = (\boldsymbol{k}+\boldsymbol{e}/2)2^{-\boldsymbol{n}};\ \boldsymbol{k}{\in}\mathbb{Z}^d;\ \boldsymbol{z}_{\boldsymbol{n},\boldsymbol{k}}{\in}\Omega\}$. Other grids are obtained by considering the *cell vertices* or *vertex nodes* of the cells in $\Omega_{\boldsymbol{n}}$ as a grid of points. We denote these grids by $\Omega^+_{\boldsymbol{n}}$.

Apparently, all grids are identified by a multi-integer $\boldsymbol{n}$; the number $|\boldsymbol{n}|$ is called the *level* of the grid $\boldsymbol{n}$. Notice that –different from classical multigrid theory– we make a clear distinction between the grid-identification $\boldsymbol{n}$ and the level number $|\boldsymbol{n}|$.

It is clear that all cells in the grids $\Omega_{\boldsymbol{n}}$ are nested in some way. Therefore, also piecewise polynomial spaces defined on these cells are nested. In the following sections we will formalise this.

We also use the following notation for partial derivatives, with $\boldsymbol{n}{\in}\mathbb{N}_0^d$,

$$D^{\boldsymbol{n}} = D^{n_1,\cdots,n_d} = \left(\frac{\partial}{\partial x_1}\right)^{n_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{n_d} .$$

For the Banach spaces of continuously differentiable functions we use, with $\boldsymbol{n}{\in}\mathbb{N}_0^d$, the notation

$$C^{\boldsymbol{n}}(\Omega) = \left\{ u \;\middle|\; \max_{\boldsymbol{0}\le\boldsymbol{m}\le\boldsymbol{n}} \max_{\boldsymbol{x}\in\Omega} |D^{\boldsymbol{m}}u(\boldsymbol{x})| < \infty \right\} ,$$

with norm

$$\|u\|_{C^{\boldsymbol{n}}} = \max_{\boldsymbol{0}\le\boldsymbol{m}\le\boldsymbol{n}} \max_{\boldsymbol{x}\in\Omega} |D^{\boldsymbol{m}}u(\boldsymbol{x})| .$$

For $l{\in}\mathbb{N}_0$, we introduce the notation $\mathcal{C}^{\boldsymbol{n},l}(\Omega) = \bigcap_{|\boldsymbol{m}|=l} C^{\boldsymbol{n}+\boldsymbol{m}}(\Omega)$. This is a generalisation which combines $C^{\boldsymbol{n}}(\Omega) = \mathcal{C}^{\boldsymbol{n},0}(\Omega)$ with the usual space of l times continuously differentiable functions $C^l(\Omega) = \mathcal{C}^{\boldsymbol{0},l}(\Omega)$. With a $C_0^{\boldsymbol{n}}(\Omega)$ and $\mathcal{C}_0^{\boldsymbol{n},l}(\Omega)$ we denote the corresponding subspaces with homogeneous boundary conditions.

For the Banach spaces of integrable functions, $1 \le p \le \infty$, we, similarly, use the notation

$$W_p^{\boldsymbol{n}}(\Omega) = \left\{ u \;\middle|\; \sum_{\boldsymbol{0}\le\boldsymbol{m}\le\boldsymbol{n}} \int_{x\in\Omega} |D^{\boldsymbol{m}}u(x)|^p < \infty \right\} .$$

and for the semi-norm and norm

$$|u|_{W_p^{\boldsymbol{n}}} = \sqrt[p]{\int_{x\in\Omega} |D^{\boldsymbol{n}}u(x)|^p} \quad \text{and} \quad \|u\|_{W_p^{\boldsymbol{n}}} = \sqrt[p]{\sum_{\boldsymbol{0}\le\boldsymbol{m}\le\boldsymbol{n}} |u|^p_{W_p^{\boldsymbol{n}}}} .$$

or, with $0 \le k \le d$,

$$\|u\|_{W_p^{\boldsymbol{n}},k} = \left(\sum_{\boldsymbol{0}\le\boldsymbol{m}\le\boldsymbol{e},|\boldsymbol{m}|=k} \|D^{\boldsymbol{e}+\boldsymbol{m}}u\|_p^p \right)^{1/p} .$$

For $l{\in}\mathbb{N}_0$, we write $\mathcal{W}_p^{\boldsymbol{n},l}(\Omega) = \bigcap_{|\boldsymbol{m}|=l} W_p^{\boldsymbol{n}+\boldsymbol{m}}(\Omega)$ and we obtain the Sobolev space $W_p^l(\Omega) = \mathcal{W}_p^{\boldsymbol{0},l}(\Omega)$. Again, with a $W_{p,0}^l(\Omega)$ and $\mathcal{W}_{p,0}^{\boldsymbol{n},l}(\Omega)$ we denote the corresponding subspaces with homogeneous boundary conditions. For $p = \infty$ we use the standard modifications, and for W_2 we also write H.

Thus, for the Hilbert spaces of square integrable functions we use the notation $H^{\boldsymbol{n}}(\Omega) = W_2^{\boldsymbol{n}}(\Omega)$, and for the semi-norm and norm $|u|_{H^{\boldsymbol{n}}} = |u|_{W_2^{\boldsymbol{n}}}$ and $\|u\|_{H^{\boldsymbol{n}}} = \|u\|_{W_2^{\boldsymbol{n}}}$ For $l{\in}\mathbb{N}_0$, we write $\mathcal{H}^{\boldsymbol{n},l}(\Omega) = \bigcap_{|\boldsymbol{m}|=l} H^{\boldsymbol{n}+\boldsymbol{m}}(\Omega)$ and we obtain the usual Sobolev space $H^l(\Omega) = \mathcal{H}^{\boldsymbol{0},l}(\Omega)$. Again, with a $H_0^l(\Omega)$ and $\mathcal{H}_0^{\boldsymbol{n},l}(\Omega)$ we denote the corresponding subspaces with homogeneous boundary conditions.

2. Space decomposition

2.1. Nested restrictions and prolongations

Let X be a Banach space; e.g. $X = C^0(\Omega)$, $X = L_p(\Omega)$ or $X = L_p^{\text{loc}}(\Omega)$, where $\Omega \subset \mathbb{R}^d$. Let $\boldsymbol{k}\in\mathbb{Z}^d$ and let

$$R_{\boldsymbol{k}} : X \to V_{\boldsymbol{k}} \tag{2.1}$$

be a *restriction*, i.e. a linear surjection. Possibly $V_{\boldsymbol{k}} \subset X$, but this is not necessary. We notice that for any such $R_{\boldsymbol{k}}$, because of the surjection, there exists the right-inverse or *reconstruction*

$$P_{\boldsymbol{k}} : V_{\boldsymbol{k}} \to X\,, \tag{2.2}$$

such that

$$R_{\boldsymbol{k}} P_{\boldsymbol{k}} = I_{\boldsymbol{k}} \tag{2.3}$$

is the identity operator on $V_{\boldsymbol{k}}$. We notice that $P_{\boldsymbol{k}}$ is an injection (and hence a prolongation) and $\text{Ran}(P_{\boldsymbol{k}}) \subset X$, but $P_{\boldsymbol{k}}$ is *not* uniquely determined by a given $R_{\boldsymbol{k}}$. In this section we study properties of such sets of *transfer operators* $\{R_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ and $\{P_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$.

It is a consequence of (2.3) that

$$\Pi_{\boldsymbol{k}} = P_{\boldsymbol{k}} R_{\boldsymbol{k}}$$

is a projection

$$\Pi_{\boldsymbol{k}} : X \to \text{Ran}(\Pi_{\boldsymbol{k}}) = \text{Ran}(P_{\boldsymbol{k}}) \subset X,$$

as is

$$I - \Pi_{\boldsymbol{k}} : X \to \text{Ker}(\Pi_{\boldsymbol{k}}) = \text{Ker}(R_{\boldsymbol{k}}) \subset X,$$

and we observe that X can be written as a direct sum $X = \text{Ran}(P_{\boldsymbol{k}}) \oplus \text{Ker}(R_{\boldsymbol{k}})$.

Definition 2.1
A set $\{R_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is called a *nested set of restrictions* (or *NSR*) iff

$$\boldsymbol{k} \geq \boldsymbol{m} \Rightarrow \text{Ker}(R_{\boldsymbol{k}}) \subset \text{Ker}(R_{\boldsymbol{m}})\,. \tag{2.4}$$

A set $\{P_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is called a *nested set of prolongations* (or *NSP*) iff

$$\boldsymbol{k} \geq \boldsymbol{m} \Rightarrow \text{Ran}(P_{\boldsymbol{k}}) \supset \text{Ran}(P_{\boldsymbol{m}})\,. \tag{2.5}$$

Remark:
It is obvious that for an NSR $\{R_{\boldsymbol{k}}\}$ a set of corresponding reconstructions is not necessarily an NSP. On the other hand, given an NSP, the corresponding set of restrictions is not necessarily an NSR. However, in some cases both the restrictions and their reconstructions may form nested sets. Then we say that the *transfer operators are nested* and $\{V_{\boldsymbol{k}}\}$ forms a nested set of representations of functions in X.

Theorem 2.2 Let $\{R_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ be a nested set of restrictions, then

$$\forall \boldsymbol{n} \geq \boldsymbol{m}\ \exists!\ R_{\boldsymbol{mn}} : V_{\boldsymbol{n}} \to V_{\boldsymbol{m}}\,,$$

with the properties:

$$\begin{array}{ll} (1) & R_{\boldsymbol{mn}} \text{ is a restriction;} \\ (2) & R_{\boldsymbol{mn}} R_{\boldsymbol{n}} = R_{\boldsymbol{m}}\,; \\ (3) & R_{\boldsymbol{mn}} = R_{\boldsymbol{m}} P_{\boldsymbol{n}} \text{ (independent of the choice of } P_{\boldsymbol{n}}!). \end{array} \tag{2.6}$$

Proof: (i) Define $R^1_{mn} = R_m P^1_n$ and $R^2_{mn} = R_m P^2_n$. Then we know that $R_n P^1_n = I_n = R_n P^2_n$ and hence $\forall\, v_n \in V_n \quad R_n(P^1_n - P^2_n)v_n = 0$. Because $\{R_k\}$ is an NSR and $m \leq n$ it follows that $R_m(P^1_n - P^2_n)v_n = 0$ and hence $R^1_{mn} = R^2_{mn}$. So that there exists a unique R_{mn}. This means that we can write $R_{mn} = R_m P_n$, and R_{mn} is independent of the choice of P_n.

(ii) $R_{mn} R_n = R_m P_n R_n = R_m \Pi_n = R_m$ on $\mathrm{Ran}(P_n)$. Now, because $X = \mathrm{Ran}(P_n) \oplus \mathrm{Ker}(R_n)$ we may write $\forall v \in X : \ v = v_p + v_n$ so that $R_{mn} R_n v = R_{mn} R_n v_p + R_{mn} R_n v_n = R_m v_p + 0 = R_m v_p$. Further, because of $\mathrm{Ker}(R_n) \subset \mathrm{Ker}(R_m)$ we see $R_m v = R_m v_p + R_m v_n = R_m v_p + 0 = R_m v_p$, so that $R_{mn} R_n v = R_m v_p = R_m v \ \forall v \in X$ and hence $R_{mn} R_n = R_m$.

(iii) Because R_m is a surjection, and by (2.6), R_{mn} is necessarily a surjection. Of course, R_{mn} is linear (trivial). Therefore R_{mn} is a restriction. □

Given an NSR $\{R_m\}_{m \in \mathbb{Z}^d}$, and a set of corresponding reconstructions $\{P_m\}_{m \in \mathbb{Z}^d}$, we introduce, for $m \leq n$,

$$P_{nm} = R_n P_m : \ V_m \to V_n \,. \tag{2.7}$$

Notice that there are many possible choices of P_m for a give R_m. Of course, some actual properties of P_{nm} may depend on this choice!

LEMMA 2.3 P_{nm} is a right inverse of R_{nm}:

$$R_{mn} P_{nm} = I_m \,. \tag{2.8}$$

Proof: $R_{mn} P_{nm} = R_m P_n R_n P_m = R_m P_m - R_m(I - P_n R_n) P_m = I_m + 0$, because $\mathrm{Ker}(R_n) \subset \mathrm{Ker}(R_m)$. □

COROLLARY 2.4
With $n \geq m$:

1. P_{nm} is a prolongation (i.e. a linear injection);
2. $P_{nm} R_{mn}$ is a projection $V_n \to \mathrm{Ran}(P_{nm}) \subset V_n$;
3. $I_n - P_{nm} R_{mn}$ is a projection $V_n \to \mathrm{Ker}(R_{mn}) \subset V_n$;
4. $V_n = \mathrm{Ran}(P_{nm}) \oplus \mathrm{Ker}(R_{mn})$;
5. $P_{nm} : V_m \to \mathrm{Ran}(P_{nm}) \subset V_n$ is a bijection.

LEMMA 2.5 Let $\{R_k\}_{k \in \mathbb{Z}^d}$ be an NSR and let $k \geq n \geq l$, then with a given set of corresponding reconstructions $\{P_k\}_{k \in \mathbb{Z}^d}$ we have

$$P_{kl} = P_{kn} P_{nl} \,.$$

Proof: $P_{kn} P_{nl} = R_k P_n R_n P_l = R_k P_l - R_k(I - P_n R_n) P_l = R_k P_l - R_k(I - \Pi_n) P_l = R_k P_l - 0 = R_k P_l$, because $\mathrm{Ker}(R_k) \subset \mathrm{Ker}(\Pi_n)$. Hence $P_{kl} = R_k P_l = P_{kn} P_{nl}$. □

LEMMA 2.6 Let $\{R_k\}_{k \in \mathbb{Z}^d}$ be an NSR with the corresponding $\{P_k\}_{k \in \mathbb{Z}^d}$ an NSP, then

$$m \geq l \Rightarrow P_m P_{ml} = P_l \,.$$

Proof: $P_m P_{ml} = P_m R_m P_l = \Pi_m P_l = P_l$. The last equality holding because $\Pi_m u = u$ for all $u \in \mathrm{Ran}(\Pi_m) = \mathrm{Ran}(P_m) \supset \mathrm{Ran}(P_l)$ □

LEMMA 2.7 If $\{R_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is an NSR, then

$$\boldsymbol{m} \geq \boldsymbol{l} \Rightarrow \Pi_{\boldsymbol{l}}\Pi_{\boldsymbol{m}} = \Pi_{\boldsymbol{l}} \,.$$

If, in addition, $\{P_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is an NSP, then also

$$\boldsymbol{m} \geq \boldsymbol{l} \Rightarrow \Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{l}} = \Pi_{\boldsymbol{l}} \,.$$

Proof: The first equality follows by $\Pi_{\boldsymbol{l}}\Pi_{\boldsymbol{m}} = P_{\boldsymbol{l}}R_{\boldsymbol{l}}P_{\boldsymbol{m}}R_{\boldsymbol{m}} = P_{\boldsymbol{l}}R_{\boldsymbol{lm}}R_{\boldsymbol{m}} = P_{\boldsymbol{l}}R_{\boldsymbol{l}} = \Pi_{\boldsymbol{l}}$, and the second equality by $\Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{l}} = P_{\boldsymbol{m}}R_{\boldsymbol{m}}P_{\boldsymbol{l}}R_{\boldsymbol{l}} = P_{\boldsymbol{m}}R_{\boldsymbol{m}}P_{\boldsymbol{m}}P_{\boldsymbol{ml}}R_{\boldsymbol{lm}}R_{\boldsymbol{m}} = P_{\boldsymbol{m}}P_{\boldsymbol{ml}}R_{\boldsymbol{lm}}R_{\boldsymbol{m}} = P_{\boldsymbol{l}}R_{\boldsymbol{l}} = \Pi_{\boldsymbol{l}}$. □

If $\{R_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is an NSR, and if $\boldsymbol{m} \leq \boldsymbol{n}$, then a bijection exists between $\mathrm{Ran}(P_{\boldsymbol{m}})$ and a subset of $\mathrm{Ran}(P_{\boldsymbol{n}})$. We denote this relation by $\mathrm{Ran}(P_{\boldsymbol{m}}) \preceq \mathrm{Ran}(P_{\boldsymbol{n}})$. I.e., a function that can be found in $\mathrm{Ran}(P_{\boldsymbol{m}})$, can uniquely be associated with a function in $\mathrm{Ran}(P_{\boldsymbol{n}})$. This follows because a bijection exists between $\mathrm{Ran}(P_{\boldsymbol{n}})$ and $V_{\boldsymbol{n}}$, and between $\mathrm{Ran}(P_{\boldsymbol{m}})$ and $V_{\boldsymbol{m}}$; and also a bijection exists between $V_{\boldsymbol{m}}$ and $\mathrm{Ran}(P_{\boldsymbol{nm}}) \subset V_{\boldsymbol{n}}$.

Hence, given a Banach space X with a nested set of restrictions $\{R_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$, a family of subspaces $\mathrm{Ran}(P_{\boldsymbol{n}})$ exists, with a partial ordering corresponding with the partial ordering of $\{\boldsymbol{n}\}$. This means that, although not necessarily $\mathrm{Ran}(P_{\boldsymbol{m}}) \subset \mathrm{Ran}(P_{\boldsymbol{n}})$, a partial ordering exists such that $\boldsymbol{m} \leq \boldsymbol{n} \Leftrightarrow \mathrm{Ran}(P_{\boldsymbol{m}}) \preceq \mathrm{Ran}(P_{\boldsymbol{n}}) \Leftrightarrow P_{\boldsymbol{nm}}R_{\boldsymbol{m}}\mathrm{Ran}(P_{\boldsymbol{m}}) = R_{\boldsymbol{n}}\mathrm{Ran}(P_{\boldsymbol{m}}) \subset R_{\boldsymbol{n}}\mathrm{Ran}(P_{\boldsymbol{n}})$.

If $\{P_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is an NSP, this partial ordering simply reduces to $\mathrm{Ran}(P_{\boldsymbol{m}}) \subset \mathrm{Ran}(P_{\boldsymbol{n}})$. Then, in the case that $V_{\boldsymbol{n}} \subset X$ and we take $P_{\boldsymbol{n}}$ to be the natural injection, this means that $\mathrm{Ran}(P_{\boldsymbol{n}})$ can be identified with $V_{\boldsymbol{n}}$ and $\boldsymbol{m} \leq \boldsymbol{n} \Leftrightarrow V_{\boldsymbol{m}} \subset V_{\boldsymbol{n}}$ and, thus, we find $\{V_{\boldsymbol{n}}\}$ to be a partially ordered family of subsets of X.

DEFINITION 2.8
Functions $f_{\boldsymbol{m}}\in V_{\boldsymbol{m}}$ and $f_{\boldsymbol{n}}\in V_{\boldsymbol{n}}$ are *mutually coherent* iff

$$\begin{array}{l} \exists f_{\boldsymbol{k}}\in V_{\boldsymbol{k}} \text{ with } \boldsymbol{k} \geq \boldsymbol{m},\ \boldsymbol{k} \geq \boldsymbol{n}, \\ \text{such that } f_{\boldsymbol{m}} = R_{\boldsymbol{mk}}f_{\boldsymbol{k}} \text{ and } f_{\boldsymbol{n}} = R_{\boldsymbol{nk}}f_{\boldsymbol{k}} \,. \end{array}$$

THEOREM 2.9 Let $\{R_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ be an NSR and $\{P_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ an NSP. Then, if $f_{\boldsymbol{m}}$ and $f_{\boldsymbol{n}}$ are mutually coherent, we have

$$\forall \boldsymbol{l}\in\mathbb{Z}^d \text{with}\, \boldsymbol{l} \leq \boldsymbol{m},\, \boldsymbol{l} \leq \boldsymbol{n},\ \text{ we have}\, R_{\boldsymbol{lm}}f_{\boldsymbol{m}} = R_{\boldsymbol{ln}}f_{\boldsymbol{n}} \,. \tag{2.9}$$

Moreover, under the additional condition that $\Pi_{\min(\boldsymbol{m},\boldsymbol{n})} = \Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}}$, also the reverse holds: it follows from (2.9) that $f_{\boldsymbol{m}}$ and $f_{\boldsymbol{n}}$ are mutually coherent.

Proof: $\Rightarrow$ First we assume that $f_{\boldsymbol{m}}\in V_{\boldsymbol{m}}$ and $f_{\boldsymbol{n}}\in V_{\boldsymbol{n}}$ are mutually coherent. Then $\exists f_{\boldsymbol{k}}\in V_{\boldsymbol{k}}$ with $\boldsymbol{k} \geq \boldsymbol{m}$, $\boldsymbol{k} \geq \boldsymbol{n}$ such that $f_{\boldsymbol{m}} = R_{\boldsymbol{mk}}f_{\boldsymbol{k}}$, $f_{\boldsymbol{n}} = R_{\boldsymbol{nk}}f_{\boldsymbol{k}}$, and hence $R_{\boldsymbol{lm}}f_{\boldsymbol{m}} = R_{\boldsymbol{lm}}R_{\boldsymbol{mk}}f_{\boldsymbol{k}} = R_{\boldsymbol{lk}}f_{\boldsymbol{k}} = R_{\boldsymbol{ln}}R_{\boldsymbol{nk}}f_{\boldsymbol{k}} = R_{\boldsymbol{ln}}f_{\boldsymbol{n}}$, which proves (2.9)

$\Leftarrow$ Now we assume (2.9). Let $\boldsymbol{k} = \max(\boldsymbol{m},\boldsymbol{n})$ and take $\boldsymbol{l} = \min(\boldsymbol{m},\boldsymbol{n})$. We introduce $f_{\boldsymbol{k}}$ by:

$$\begin{aligned} f_{\boldsymbol{k}} &= P_{\boldsymbol{km}}f_{\boldsymbol{m}} + P_{\boldsymbol{kn}}f_{\boldsymbol{n}} - P_{\boldsymbol{kl}}R_{\boldsymbol{lm}}f_{\boldsymbol{m}} = \\ &= P_{\boldsymbol{km}}f_{\boldsymbol{m}} + P_{\boldsymbol{kn}}f_{\boldsymbol{n}} - P_{\boldsymbol{kl}}R_{\boldsymbol{ln}}f_{\boldsymbol{n}} \,. \end{aligned}$$

Then

$$\begin{aligned} R_{\boldsymbol{mk}}f_{\boldsymbol{k}} &= R_{\boldsymbol{mk}}P_{\boldsymbol{km}}f_{\boldsymbol{m}} + R_{\boldsymbol{mk}}P_{\boldsymbol{kn}}f_{\boldsymbol{n}} - R_{\boldsymbol{mk}}P_{\boldsymbol{kl}}R_{\boldsymbol{ln}}f_{\boldsymbol{n}} \\ &= R_{\boldsymbol{mk}}P_{\boldsymbol{km}}f_{\boldsymbol{m}} + R_{\boldsymbol{mk}}P_{\boldsymbol{kn}}(I_{\boldsymbol{n}} - P_{\boldsymbol{nl}}R_{\boldsymbol{ln}})f_{\boldsymbol{n}} \,, \end{aligned}$$

Now, because $\boldsymbol{k} \geq \boldsymbol{m}$ and $\boldsymbol{k} \geq \boldsymbol{n} \geq \boldsymbol{l}$, we know $R_{\boldsymbol{mk}} P_{\boldsymbol{km}} = I_{\boldsymbol{m}}$ (Lemma 2.3).

The additional condition and Lemma (2.7) show $\Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}} = \Pi_{\boldsymbol{l}} = \Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{l}}\Pi_{\boldsymbol{n}}$, so that $\Pi_{\boldsymbol{m}}(I - \Pi_{\boldsymbol{l}})\Pi_{\boldsymbol{n}} = 0$. Hence,

$$\begin{array}{l} R_{\boldsymbol{mk}} P_{\boldsymbol{kn}} (I_{\boldsymbol{n}} - P_{\boldsymbol{nl}} R_{\boldsymbol{ln}}) = \\ R_{\boldsymbol{m}} P_{\boldsymbol{k}} R_{\boldsymbol{k}} P_{\boldsymbol{n}} (I_{\boldsymbol{n}} - R_{\boldsymbol{n}} P_{\boldsymbol{l}} R_{\boldsymbol{l}} P_{\boldsymbol{n}}) = \\ R_{\boldsymbol{m}} P_{\boldsymbol{k}} R_{\boldsymbol{k}} (I - P_{\boldsymbol{n}} R_{\boldsymbol{n}} P_{\boldsymbol{l}} R_{\boldsymbol{l}}) P_{\boldsymbol{n}} = \\ R_{\boldsymbol{m}} \Pi_{\boldsymbol{m}} \Pi_{\boldsymbol{k}} (I - \Pi_{\boldsymbol{n}} \Pi_{\boldsymbol{l}}) \Pi_{\boldsymbol{n}} P_{\boldsymbol{n}} = \\ R_{\boldsymbol{m}} \ \Pi_{\boldsymbol{m}} (I - \Pi_{\boldsymbol{l}}) \Pi_{\boldsymbol{n}} \ P_{\boldsymbol{n}} = 0 \,. \end{array}$$

Thus, we find $R_{\boldsymbol{mk}} f_{\boldsymbol{k}} = f_{\boldsymbol{m}} + 0 = f_{\boldsymbol{m}}$. Analogously we prove $R_{\boldsymbol{nk}} f_{\boldsymbol{k}} = f_{\boldsymbol{n}}$. □

2.2. Commutative restrictions and projections

DEFINITION 2.10 A set $\{R_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is called a *commutative set of restrictions* (or a *CSR*) iff for all $\boldsymbol{m}, \boldsymbol{n}\in\mathbb{Z}^d$

$$\mathrm{Ker}(R_{\min(\boldsymbol{m},\boldsymbol{n})}) = \begin{array}{c} \mathrm{Ker}(R_{\boldsymbol{n}}) \cap \mathrm{Ker}(R_{\boldsymbol{m}}) \\ \oplus \\ \mathrm{Ker}(R_{\boldsymbol{n}}) \cap \mathrm{Ran}(P_{\boldsymbol{m}}) \\ \oplus \\ \mathrm{Ran}(P_{\boldsymbol{n}}) \cap \mathrm{Ker}(R_{\boldsymbol{m}}) \end{array} . \tag{2.10}$$

A set $\{P_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is called a *commutative set of prolongations* (or a *CSP*) iff for all $\boldsymbol{m}, \boldsymbol{n}\in\mathbb{Z}^d$

$$\mathrm{Ran}(P_{\min(\boldsymbol{m},\boldsymbol{n})}) = \mathrm{Ran}(P_{\boldsymbol{n}}) \cap \mathrm{Ran}(P_{\boldsymbol{m}})\,. \tag{2.11}$$

If $\{R_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is a CSR and $\{P_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is a CSP, then we say that we have *commutative transfer operators.* .

Remark:

It is immediate that each CSR is an NSR and each CSP is an NSP: which simply follows from the equivalence $\boldsymbol{n} \leq \boldsymbol{m} \Leftrightarrow \boldsymbol{n} = \min(\boldsymbol{n}, \boldsymbol{m})$. On the other hand not necessarily every NSP is a CSP nor every NSR a CSR.

In the next theorem we show how the above definition of a CSR and an NSP lead to commutative projection operators indeed. To prove the theorem, we first derive a few lemmas.

LEMMA 2.11 If (2.12) holds, then $\{R_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is a CSR.

Proof: Let $\boldsymbol{m}, \boldsymbol{n}\in\mathbb{Z}^d$ be arbitrary, and let $\boldsymbol{l} = \min(\boldsymbol{m}, \boldsymbol{n})$, then we know by assumption $\Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}} = \Pi_{\boldsymbol{l}} = \Pi_{\boldsymbol{n}}\Pi_{\boldsymbol{m}}$. We define $M = \mathrm{Span}((\mathrm{Ker}(R_{\boldsymbol{n}}) \cap \mathrm{Ker}(R_{\boldsymbol{m}})), (\mathrm{Ker}(R_{\boldsymbol{n}}) \cap \mathrm{Ran}(P_{\boldsymbol{m}})), (\mathrm{Ran}(P_{\boldsymbol{n}}) \cap \mathrm{Ker}(R_{\boldsymbol{m}})))$. To prove the lemma we show (i) $M \subset \mathrm{Ker}(\Pi_{\boldsymbol{l}})$, and (ii) $\mathrm{Ker}(\Pi_{\boldsymbol{l}}) \subset M$, and (iii) $M = (\mathrm{Ker}(R_{\boldsymbol{n}}) \cap \mathrm{Ker}(R_{\boldsymbol{m}})) \oplus (\mathrm{Ker}(R_{\boldsymbol{n}}) \cap \mathrm{Ran}(P_{\boldsymbol{m}})) \oplus (\mathrm{Ran}(P_{\boldsymbol{n}}) \cap \mathrm{Ker}(R_{\boldsymbol{m}}))$.

To prove (i), let $x\in M$, then $x = x_m + x_n + x_l$ with $x_n\in\mathrm{Ker}(\Pi_{\boldsymbol{n}}) \cap \mathrm{Ran}(\Pi_{\boldsymbol{m}})$, $x_m\in\mathrm{Ker}(\Pi_{\boldsymbol{m}}) \cap \mathrm{Ran}(\Pi_{\boldsymbol{n}})$, $x_l\in\mathrm{Ker}(\Pi_{\boldsymbol{n}}) \cap \mathrm{Ker}(\Pi_{\boldsymbol{m}})$. Then $\Pi_{\boldsymbol{l}} x = \Pi_{\boldsymbol{n}}\Pi_{\boldsymbol{m}} x_m + \Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}} x_n + \Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}} x_l = \Pi_{\boldsymbol{n}} 0 + \Pi_{\boldsymbol{m}} 0 + \Pi_{\boldsymbol{m}} 0 = 0$. So that $x\in\mathrm{Ker}(\Pi_{\boldsymbol{l}})$.

To prove (ii), let $x\in\mathrm{Ker}(\Pi_{\boldsymbol{l}})$ be arbitrary and define $x_m := \Pi_{\boldsymbol{m}}(I - \Pi_{\boldsymbol{n}})x = \Pi_{\boldsymbol{m}} x$ and $x_n := \Pi_{\boldsymbol{n}}(I - \Pi_{\boldsymbol{m}})x = \Pi_{\boldsymbol{n}} x$ and $x_0 := (I - \Pi_{\boldsymbol{m}} - \Pi_{\boldsymbol{n}} + \Pi_{\boldsymbol{n}}\Pi_{\boldsymbol{m}})x = (I - \Pi_{\boldsymbol{m}} - \Pi_{\boldsymbol{n}})x$. Then a simple calculation shows: $x_0 \in \mathrm{Ker}(R_{\boldsymbol{n}}) \cap \mathrm{Ker}(R_{\boldsymbol{m}})$ and $x_m \in \mathrm{Ker}(R_{\boldsymbol{n}}) \cap \mathrm{Ran}(P_{\boldsymbol{m}})$ and $x_n \in \mathrm{Ran}(P_{\boldsymbol{n}}) \cap \mathrm{Ker}(R_{\boldsymbol{m}})$ and $x_0 + x_m + x_n = x$.

To prove (iii), we show that M is a direct sum of the three spanning spaces. For this we have to prove: if $x_0 \in \mathrm{Ker}(R_{\boldsymbol{n}}) \cap \mathrm{Ker}(R_{\boldsymbol{m}})$ and $x_m \in \mathrm{Ker}(R_{\boldsymbol{n}}) \cap \mathrm{Ran}(P_{\boldsymbol{m}})$ and $x_n \in \mathrm{Ran}(P_{\boldsymbol{n}}) \cap \mathrm{Ker}(R_{\boldsymbol{m}})$ and $x_0 + x_m + x_n = 0$, then $x_0 = x_m = x_n = 0$. This is seen by $0 = \Pi_{\boldsymbol{m}}(I - \Pi_{\boldsymbol{n}})(x_0 + x_m + x_n) = x_m$

and $0 = \Pi_{\boldsymbol{n}}(I - \Pi_{\boldsymbol{m}})(x_0 + x_m + x_n) = x_n$. This implies $x_0 = 0$. □

LEMMA 2.12 If (2.12) holds, then $\{P_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is a NSP.
Proof: Let $\boldsymbol{n} < \boldsymbol{m}$ then (2.12) implies $\Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}} = \Pi_{\boldsymbol{n}}$, and hence, for all $x\in X$ we have $\Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}}x = \Pi_{\boldsymbol{n}}x$. It follows that for all $x\in X$ holds $\Pi_{\boldsymbol{n}}x\in\mathrm{Ran}(\Pi_{\boldsymbol{m}})$. Therefore $\mathrm{Ran}(\Pi_{\boldsymbol{n}}) \subset \mathrm{Ran}(\Pi_{\boldsymbol{m}})$ for all $\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^d$ with $\boldsymbol{n} < \boldsymbol{m}$. Hence $\{P_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is a NSP. □

LEMMA 2.13 If $\{R_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is a CSR and $\{P_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is a NSP then (2.12) holds.
Proof: Let $\boldsymbol{l} = \min(\boldsymbol{m},\boldsymbol{n})$ and let $x\in X$ be arbitrary. We know that $X = \mathrm{Ran}(P_{\boldsymbol{l}}) \oplus \mathrm{Ker}(R_{\boldsymbol{l}}) = \mathrm{Ran}(P_{\boldsymbol{l}}) \oplus(\mathrm{Ker}(R_{\boldsymbol{n}})\cap\mathrm{Ker}(R_{\boldsymbol{m}})) \oplus(\mathrm{Ran}(P_{\boldsymbol{n}})\cap\mathrm{Ker}(R_{\boldsymbol{m}})) \oplus(\mathrm{Ker}(R_{\boldsymbol{n}})\cap\mathrm{Ran}(P_{\boldsymbol{m}}))$ because $\{R_{\boldsymbol{k}}\}$ is a CSR. Hence, we may split $x = x_l + x_o + x_m + x_n$ accordingly. Now we know $\Pi_{\boldsymbol{l}}x = x_l$ and $\Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}}x = \Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}}(x_l + x_o + x_m + x_n) = \Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}}x_l + \Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}}x_o + \Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}}x_m + \Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}}x_n = \Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}}x_l + \Pi_{\boldsymbol{m}}0 + \Pi_{\boldsymbol{m}}x_m + \Pi_{\boldsymbol{m}}0 = \Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}}x_l$. Because $\{P_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is a NSP, we know that $x_l\in\mathrm{Ran}(P_{\boldsymbol{l}}) \subset \mathrm{Ran}(P_{\boldsymbol{n}})$ and $x_l\in\mathrm{Ran}(P_{\boldsymbol{l}}) \subset \mathrm{Ran}(P_{\boldsymbol{m}})$; hence $\Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}}x_l = x_l$. We conclude that, for arbitrary $x\in X$ holds $\Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}}x = x_l = \Pi_{\boldsymbol{l}}x$; which proves the lemma. □

THEOREM 2.14 The two following statements are equivalent:
(1) $\{R_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is a CSR, and $\{P_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is an NSP; and
(2)

$$\Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}} = \Pi_{\min(\boldsymbol{m},\boldsymbol{n})} \quad \forall \boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^d\,. \tag{2.12}$$

Proof: The theorem is a direct combination of the three lemmas above. □

It is an immediate consequence of the theorem that operators $\Pi_{\boldsymbol{n}}$ associated with commutative transfer operators $\{R_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ and $\{P_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ do commute:

$$\Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}} = \Pi_{\min(\boldsymbol{m},\boldsymbol{n})} = \Pi_{\boldsymbol{n}}\Pi_{\boldsymbol{m}}\,.$$

Further, combination with Theorem 2.9 gives the following important result.

COROLLARY 2.15
Let $\{R_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ be a CSR and $\{P_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ an NSP, then $f_{\boldsymbol{m}}$ and $f_{\boldsymbol{n}}$ are mutually coherent if and only if

$$\forall \boldsymbol{l}\in\mathbb{Z}^d \text{with}\, \boldsymbol{l} \le \boldsymbol{m},\, \boldsymbol{l} \le \boldsymbol{n},\ \text{we have}\, R_{\boldsymbol{l}\boldsymbol{m}} f_{\boldsymbol{m}} = R_{\boldsymbol{l}\boldsymbol{n}} f_{\boldsymbol{n}}\,. \tag{2.13}$$

2.3. The merging operator

Now we have seen how information about a function $u\in X$ can be represented on $V_{\boldsymbol{n}}$, and how the representations $R_{\boldsymbol{n}}u$ are related for different $\boldsymbol{n}\in\mathbb{Z}^d$. An important question is how these $R_{\boldsymbol{n}}u$, given for a limited number of $\boldsymbol{n}\in\mathbb{Z}^d$, can be used to restore the picture of the original function u as complete as possible.

We start with the situation where information is available from two representations, viz. in $V_{\boldsymbol{n}}, V_{\boldsymbol{m}}$. Therefore we introduce the *merging* operator $\Pi_{\boldsymbol{m}\boldsymbol{n}}$, which selects for an $x\in X$ the information that can be represented by the combined representations in $V_{\boldsymbol{n}}$ and $V_{\boldsymbol{m}}$.

DEFINITION 2.16 The *merging operator* $\Pi_{\boldsymbol{m}\boldsymbol{n}} : X \to X$ is defined by

$$\Pi_{\boldsymbol{m}\boldsymbol{n}} = \Pi_{\boldsymbol{m}} + \Pi_{\boldsymbol{n}} - \Pi_{\min(\boldsymbol{m},\boldsymbol{n})}\,. \tag{2.14}$$

LEMMA 2.17 Let $\{R_{\boldsymbol{n}}\}$ and $\{P_{\boldsymbol{n}}\}$ be a set of commutative transfer operators, then

1. $$\Pi_{\boldsymbol{mn}} = \Pi_{\boldsymbol{nm}}. \tag{2.15}$$

2. $$\text{if } \boldsymbol{m} \le \boldsymbol{n} \text{ then } \Pi_{\boldsymbol{mn}} = \Pi_{\boldsymbol{n}}. \tag{2.16}$$

3. $$\Pi_{\boldsymbol{mn}} \text{ is a projection,} \tag{2.17}$$

4. $$\mathrm{Ran}(\Pi_{\boldsymbol{mn}}) = \begin{array}{c} \mathrm{Ran}(P_{\boldsymbol{n}}) \cap \mathrm{Ran}(P_{\boldsymbol{m}}) \\ \oplus \\ \mathrm{Ran}(P_{\boldsymbol{n}}) \cap \mathrm{Ker}(R_{\boldsymbol{m}}) \\ \oplus \\ \mathrm{Ker}(R_{\boldsymbol{n}}) \cap \mathrm{Ran}(P_{\boldsymbol{m}}) \end{array}. \tag{2.18}$$

5. $$\mathrm{Ker}(\Pi_{\boldsymbol{mn}}) = \mathrm{Ker}(R_{\boldsymbol{n}}) \cap \mathrm{Ker}(R_{\boldsymbol{m}}) \tag{2.19}$$

Proof: The first two statements are trivial by the definition of $\Pi_{\boldsymbol{mn}}$. Now set $\boldsymbol{l} = \min(\boldsymbol{m}, \boldsymbol{n})$.
• Because $\Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}} = \Pi_{\boldsymbol{n}}\Pi_{\boldsymbol{m}} = \Pi_{\min(\boldsymbol{m},\boldsymbol{n})}$ we have

$$\begin{aligned} \Pi_{\boldsymbol{mn}}\Pi_{\boldsymbol{mn}} &= (\Pi_{\boldsymbol{m}} + \Pi_{\boldsymbol{n}} - \Pi_{\min(\boldsymbol{m},\boldsymbol{n})})(\Pi_{\boldsymbol{m}} + \Pi_{\boldsymbol{n}} - \Pi_{\min(\boldsymbol{m},\boldsymbol{n})}) \\ &= \Pi_{\boldsymbol{m}} + \Pi_{\boldsymbol{l}} - \Pi_{\boldsymbol{l}} + \Pi_{\boldsymbol{l}} + \Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{l}} - \Pi_{\boldsymbol{l}} - \Pi_{\boldsymbol{l}} + \Pi_{\boldsymbol{l}} \\ &= \Pi_{\boldsymbol{m}} + \Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{l}} = \Pi_{\boldsymbol{mn}}\,. \end{aligned}$$

Hence, $\Pi_{\boldsymbol{mn}}$ is a projection.
• x$\in$Ran($\Pi_{\boldsymbol{mn}}$) implies $\exists z \quad x = \Pi_{\boldsymbol{mn}}z = \Pi_{\boldsymbol{m}}z + \Pi_{\boldsymbol{n}}z - \Pi_{\boldsymbol{l}}z = (\Pi_{\boldsymbol{m}} - \Pi_{\boldsymbol{l}})z + (\Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{l}})z + \Pi_{\boldsymbol{l}}z = z_m + z_n + z_l$. It is clear that this is a decomposition according to the direct sum in (2.18): $z_l \in \mathrm{Ran}(\Pi_{\boldsymbol{l}}) \subset \mathrm{Ran}(\Pi_{\boldsymbol{m}}) \cap \mathrm{Ran}(\Pi_{\boldsymbol{n}})$; $z_m \in \mathrm{Ran}(\Pi_{\boldsymbol{m}})$ and $z_m \in \mathrm{Ker}(\Pi_{\boldsymbol{n}})$ because $\Pi_{\boldsymbol{n}}z_m = \Pi_{\boldsymbol{n}}(\Pi_{\boldsymbol{m}} - \Pi_{\boldsymbol{l}}) = \Pi_{\min(\boldsymbol{n},\boldsymbol{m})} - \Pi_{\boldsymbol{l}}z = 0$; and similarly $z_n \in \mathrm{Ran}(\Pi_{\boldsymbol{n}}) \cap \mathrm{Ker}(\Pi_{\boldsymbol{m}})$.
• On the other hand, if $z = z_l + z_n + z_m$ is a splitting according to the direct sum, then $\Pi_{\boldsymbol{l}}z = \Pi_{\boldsymbol{l}}(z_l + z_m + z_n) = \Pi_{\boldsymbol{l}}z_l + \Pi_{\boldsymbol{n}}\Pi_{\boldsymbol{m}}\boldsymbol{z}_m + \Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}}z_n = z_l + \Pi_{\boldsymbol{n}}z_m + \Pi_{\boldsymbol{m}}z_n = z_l$ and $(\Pi_{\boldsymbol{m}} - \Pi_{\boldsymbol{l}})z = (\Pi_{\boldsymbol{m}} - \Pi_{\boldsymbol{l}})(z_l + z_m + z_n) = (\Pi_{\boldsymbol{m}} - \Pi_{\boldsymbol{l}})z_l + (\Pi_{\boldsymbol{m}} - \Pi_{\boldsymbol{l}})z_m + (\Pi_{\boldsymbol{m}} - \Pi_{\boldsymbol{l}})z_n = z_l - z_l + \Pi_{\boldsymbol{m}}(I - \Pi_{\boldsymbol{n}})z_m + \Pi_{\boldsymbol{m}}(I - \Pi_{\boldsymbol{n}})z_n = \Pi_{\boldsymbol{m}}z_m = z_m$. Analogously $(\Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{l}})z = z_n$. Thus $z = z_l + z_m + z_n = \Pi_{\boldsymbol{l}}z + (\Pi_{\boldsymbol{m}} - \Pi_{\boldsymbol{l}})z + (\Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{l}})z = (\Pi_{\boldsymbol{m}} + \Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{l}})z = \Pi_{\boldsymbol{mn}}z$. Hence $z \in \mathrm{Ran}(\Pi_{\boldsymbol{mn}})$.
• Assume that $0 = z = z_l + z_n + z_m$ is a splitting as above, then it follows that $0 = z_l = z_n = z_m$ because $0 = \Pi_{\boldsymbol{l}}(z) = (\Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}})(z_l + z_n + z_m) = z_l$ and $0 = (\Pi_{\boldsymbol{m}} - \Pi_{\boldsymbol{l}})(z) = (I - \Pi_{\boldsymbol{n}})\Pi_{\boldsymbol{m}}(z_l + z_n + z_m) = z_m$ and $0 = (\Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{l}})(z) = (I - \Pi_{\boldsymbol{m}})\Pi_{\boldsymbol{n}}(z_l + z_n + z_m) = z_n$.
• $\Pi_{\boldsymbol{mn}} = P_{\boldsymbol{m}}R_{\boldsymbol{m}} + P_{\boldsymbol{n}}R_{\boldsymbol{n}} - P_{\boldsymbol{l}}R_{\boldsymbol{l}}$, hence $\mathrm{Ker}(\Pi_{\boldsymbol{mn}}) \supset \mathrm{Ker}(R_{\boldsymbol{n}}) \cap \mathrm{Ker}(R_{\boldsymbol{m}})$ is trivial.
• $\mathrm{Ker}(\Pi_{\boldsymbol{mn}}) \subset \mathrm{Ker}(R_{\boldsymbol{n}}) \cap \mathrm{Ker}(R_{\boldsymbol{m}})$ is shown as follows. Let $x \in \mathrm{Ker}(\Pi_{\boldsymbol{nm}})$, then $0 = \Pi_{\boldsymbol{nm}}x = (\Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{l}})x + (\Pi_{\boldsymbol{m}} - \Pi_{\boldsymbol{l}})x + \Pi_{\boldsymbol{l}}x =: x_n + x_m + x_l$. This implies $x_l \in \mathrm{Ran}(\Pi_{\boldsymbol{l}})$ and $x_n, x_m \in \mathrm{Ker}(\Pi_{\boldsymbol{l}})$, so that $x_l = 0$ and $x_n + x_m = 0$. Further, from $x_n = -x_m \in \mathrm{Ran}(\Pi_{\boldsymbol{m}}) \cap \mathrm{Ran}(\Pi_{\boldsymbol{n}}) = \mathrm{Ran}(\Pi_{\min(\boldsymbol{n},\boldsymbol{m})}) = \mathrm{Ran}(P_{\boldsymbol{l}})$ it follows that $x_m = x_n = 0$, and hence $\mathrm{Ker}(\Pi_{\boldsymbol{mn}}) = \mathrm{Ker}(R_{\boldsymbol{n}}) \cap \mathrm{Ker}(R_{\boldsymbol{m}})$. □

2.4. The hierarchical decomposition

Further in this section we assume commutative transfer operators.

DEFINITION 2.18 Let $\{R_{\boldsymbol{n}}\}$ and $\{P_{\boldsymbol{n}}\}$ be a commutative set of transfer operators, then we define $H_{\boldsymbol{mn}} : X \to \mathrm{Ran}(P_{\max(\boldsymbol{m},\boldsymbol{n})}) \subset X$, the *hierarchical surplus, relative to* the grids $\boldsymbol{m}$ and $\boldsymbol{n}$, by

$$H_{\boldsymbol{mn}} = \Pi_{\max(\boldsymbol{m},\boldsymbol{n})} - \Pi_{\boldsymbol{mn}} \,. \tag{2.20}$$

Remarks:

- The hierarchical surplus is a projection operator.
- The hierarchical surplus, $H_{\boldsymbol{mn}}u$, of a function u represents the amount of information in $u \in V_{\max(\boldsymbol{m},\boldsymbol{n})}$ that cannot be represented on the $\mathrm{Span}(\mathrm{Ran}(P_{\boldsymbol{n}}), \mathrm{Ran}(P_{\boldsymbol{m}}))$.
- We can also write

$$H_{\boldsymbol{mn}} = \left(\Pi_{\max(\boldsymbol{n},\boldsymbol{m})} - \Pi_{\boldsymbol{n}}\right)\left(\Pi_{\max(\boldsymbol{n},\boldsymbol{m})} - \Pi_{\boldsymbol{m}}\right) \,.$$

DEFINITION 2.19 For a fixed $\boldsymbol{n_0} \in \{-\infty, \mathbb{Z}\}^d$, which indicates a coarsest grid, we define for arbitrary $\boldsymbol{n} \geq \boldsymbol{n_0}$, $\boldsymbol{n} \in \mathbb{Z}^d$, the operator $Q_{\boldsymbol{n}} : X \to \mathrm{Ran}(P_{\boldsymbol{n}}) \subset X$, by

$$Q_{\boldsymbol{n}} = \prod_{\substack{j=1 \\ n_j \neq n_{0j}}}^{d} \left(\Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{n}-\boldsymbol{e}_j}\right) \,. \tag{2.21}$$

We use the convention that $Q_{\boldsymbol{n_0}} = \Pi_{\boldsymbol{n_0}}$. The operator $Q_{\boldsymbol{n}}$ is called the *(direct) hierarchical surplus* at grid $\boldsymbol{n}$.

If $\boldsymbol{n_0} \in \mathbb{Z}^d$ we call $\boldsymbol{n_0} \in \mathbb{Z}^d$ the *coarsest grid*, and, without loss of generality we may assume $\boldsymbol{n_0} = \boldsymbol{0}$. If a $n_{0j} = -\infty$ then no coarsest grid exists.

LEMMA 2.20 $Q_{\boldsymbol{n}}$ is a projection and $Q_{\boldsymbol{m}}Q_{\boldsymbol{n}} = 0$ for all $\boldsymbol{m} \neq \boldsymbol{n}$.

Proof: First we show that $Q_{\boldsymbol{n}}$ is a projection. For simplicity of notation, we set $C = n_{\boldsymbol{0}}$.

$$\begin{aligned} Q_{\boldsymbol{n}}Q_{\boldsymbol{n}} &= \textstyle\prod_{j=1,n_j\neq c_j}^d \left(\Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{n}-\boldsymbol{e}_j}\right)\left(\Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{n}-\boldsymbol{e}_j}\right) \\ &= \textstyle\prod_{j=1,n_j\neq c_j}^d \left(\Pi_{\boldsymbol{n}}\Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{n}-\boldsymbol{e}_j}\Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{n}}\Pi_{\boldsymbol{n}-\boldsymbol{e}_j} + \Pi_{\boldsymbol{n}-\boldsymbol{e}_j}\Pi_{\boldsymbol{n}-\boldsymbol{e}_j}\right) \\ &= \textstyle\prod_{j=1,n_j\neq c_j}^d \left(\Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{n}-\boldsymbol{e}_j} - \Pi_{\boldsymbol{n}-\boldsymbol{e}_j} + \Pi_{\boldsymbol{n}-\boldsymbol{e}_j}\right) \\ &= \textstyle\prod_{j=1,n_j\neq c_j}^d \left(\Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{n}-\boldsymbol{e}_j}\right) \quad = \quad Q_{\boldsymbol{n}} \end{aligned} \tag{2.22}$$

To show that $Q_{\boldsymbol{m}}Q_{\boldsymbol{n}} = 0$ for all $\boldsymbol{m} \neq \boldsymbol{n}$, let $\boldsymbol{m} \neq \boldsymbol{n}$. Without loss of generality we may assume $n_i < m_i$ for some $i \in \{1, \ldots, d\}$. We consider the case $n_i \neq c_i$; the other cases are similar.

$$\begin{aligned} & Q_{\boldsymbol{m}}Q_{\boldsymbol{n}} \\ =\; & \textstyle\prod_j \left(\Pi_{\boldsymbol{m}} - \Pi_{\boldsymbol{m}-\boldsymbol{e}_j}\right)\left(\Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{n}-\boldsymbol{e}_j}\right) \\ =\; & \left(\Pi_{\boldsymbol{m}} - \Pi_{\boldsymbol{m}-\boldsymbol{e}_i}\right)\left(\Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{n}-\boldsymbol{e}_i}\right)\textstyle\prod_{j\neq i}\cdots \\ =\; & \left(\Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{m}-\boldsymbol{e}_i}\Pi_{\boldsymbol{n}} - \Pi_{\boldsymbol{m}}\Pi_{\boldsymbol{n}-\boldsymbol{e}_i} + \Pi_{\boldsymbol{m}-\boldsymbol{e}_i}\Pi_{\boldsymbol{n}-\boldsymbol{e}_i}\right)\textstyle\prod_{j\neq i}\cdots \\ =\; & \left(\Pi_{(\ldots,n_l,\ldots)} - \Pi_{(\ldots,n_l,\ldots)} - \Pi_{(\ldots,n_l-1,\ldots)} + \Pi_{(\ldots,n_l-1,\ldots)}\right)\textstyle\prod_{j\neq i}\cdots \\ =\; & 0 \cdot \textstyle\prod_{j\neq i}\cdots = 0 \end{aligned} \tag{2.23}$$

The indices indicated by dots correspond with those of $\min(\boldsymbol{n}, \boldsymbol{m})$. □

Remark:
Notice that, for $\boldsymbol{n} > \mathbb{N}_{\mathbf{0}}$, the two-dimensional case the relation (2.21) reads

$$Q_{\boldsymbol{n}}u = \Pi_{\boldsymbol{n}}u - \Pi_{\boldsymbol{n}-\boldsymbol{e}_1}u - \Pi_{\boldsymbol{n}-\boldsymbol{e}_2}u + \Pi_{\boldsymbol{n}-\boldsymbol{e}}u\,, \tag{2.24}$$

where $\boldsymbol{e} = (1,1)$, and in the one-dimensional case we have

$$Q_{\boldsymbol{n}}u = \Pi_{\boldsymbol{n}}u - \Pi_{\boldsymbol{n}-\boldsymbol{e}}u\,. \tag{2.25}$$

Corollary 2.21
From Definition 2.19 it is immediately obvious that the projection $\Pi_{\boldsymbol{n}}$ can be decomposed as

$$\Pi_{\boldsymbol{n}} = \sum_{\boldsymbol{n}_{\mathbf{0}} \le \boldsymbol{m} \le \boldsymbol{n}} Q_{\boldsymbol{m}}\,, \tag{2.26}$$

and, hence, $\mathrm{Ran}(\Pi_{\boldsymbol{n}}) = \mathrm{Span}(\mathrm{Ran}(Q_{\boldsymbol{m}}))_{\boldsymbol{n}_{\mathbf{0}} \le \boldsymbol{m} \le \boldsymbol{n}}$. Because of Lemma 2.20 we can write

$$\mathrm{Ran}(\Pi_{\boldsymbol{n}}) = \bigoplus_{\boldsymbol{n}_{\mathbf{0}} \le \boldsymbol{m} \le \boldsymbol{n}} \mathrm{Ran}(Q_{\boldsymbol{m}})\,. \tag{2.27}$$

If $V_{\boldsymbol{n}} \subset X$ and $P_{\boldsymbol{n}}$ is the natural injection, then we see that $\mathrm{Ran}(P_{\boldsymbol{n}}) = \mathrm{Ran}(\Pi_{\boldsymbol{n}}) = V_{\boldsymbol{n}}$ and, defining the *pre-wavelet space* $W_{\boldsymbol{n}} = \mathrm{Ran}(Q_{\boldsymbol{m}})$, we find

$$V_{\boldsymbol{n}} = \bigoplus_{\boldsymbol{n}_{\mathbf{0}} \le \boldsymbol{m} \le \boldsymbol{n}} W_{\boldsymbol{n}}\,. \tag{2.28}$$

2.5. The tensor product case
For any $i = 1, \ldots, d$, let $\Omega_i \subset \mathbb{R}$ and let $\Omega = \otimes_{i=1}^d \Omega_i \subset \mathbb{R}^d$ be their Cartesian product. Let $X_i(\Omega_i)$ be a function space on Ω_i with functions $u_{i,a_i}(x_i)$ so that $X_i(\Omega_i) = \mathrm{Span}(\{u_{i,a_i}\}_{a_i \in A_i})$. Let $A = \otimes_{i=1}^d A_i$ be the Cartesian product of index sets. Then the *tensor product space* $X(\Omega)$ is defined by

$$X(\Omega) = \otimes_{i=1}^d X_i(\Omega_i) = \mathop{\mathrm{Span}}_{\boldsymbol{a} \in A} \prod_{i=1}^d u_{i,a_i}\,. \tag{2.29}$$

It is well known , e.g., that for $X_i(\Omega_i) = C_0^\infty(\mathbb{R})$, the tensor product space $X(\Omega)$ is densely embedded both in $C^l(\mathbb{R}^d)$ and in $H^l(\mathbb{R}^d)$.

For each $i = 1, \ldots, d$, let $\{R_{i,n}\}_{n \in \mathbb{Z}}$ be a sequence of restrictions (for functions in one dimension) defined on $X_i(\Omega_i)$, with $R_{i,n} : X_i(\Omega_i) \to V_{i,n}(\Omega_i) \subset X_i(\Omega_i)$, then, similar to $X(\Omega)$, for each $\boldsymbol{n} \in \mathbb{Z}^d$ we may define a tensor product space $V_{\boldsymbol{n}}(\Omega) = \otimes_{i=1}^d V_{i,n_i}(\Omega_i)$.

If the elements of $V_{n_i}(\Omega_i)$ are all determined by values associated with Ω_{n_i} ($\Omega_{n_i}^+$ or $\Omega_{n_i}^*$), then we find a bijection $V_{\boldsymbol{n}} = V_{\boldsymbol{n}}(\Omega) \cong V_{\boldsymbol{n}}(\Omega_{\boldsymbol{n}})$, (or $\cong V_{\boldsymbol{n}}(\Omega_{\boldsymbol{n}}^+)$ or $\cong V_{\boldsymbol{n}}(\Omega_{\boldsymbol{n}}^*)$). This notation indicates that its elements are determined by their values on the Cartesian product space $\Omega_{\boldsymbol{n}} = \otimes_{i=1}^d \Omega_{n_i}$ (or $\Omega_{\boldsymbol{n}}^+$ or $\Omega_{\boldsymbol{n}}^*$).

Definition 2.22 We define the *tensor product restriction* $R_{\boldsymbol{n}} : X(\Omega) \to V_{\boldsymbol{n}}(\Omega)$ by its action on a typical basis function

$$u_{\boldsymbol{a}}(\boldsymbol{x}) = \prod_{i=1}^d u_{i,a_i}(x_i) \mapsto R_{\boldsymbol{n}} u_{\boldsymbol{a}}(\boldsymbol{x}) = \prod_{i=1}^d R_{i,n_i} u_{i,a_i}(x_i)\,. \tag{2.30}$$

We also write $R_{\boldsymbol{n}} = \otimes_i R_{i,n_i}$. Since $V_{\boldsymbol{n}}(\Omega) \subset X(\Omega)$, we can take the natural injection $P_{\boldsymbol{n}}$ as the corresponding reconstruction for $R_{\boldsymbol{n}}$. This $P_{\boldsymbol{n}}$ we call the *tensor product prolongation.*

THEOREM 2.23 For each $i \in \{1,...,d\}$ let $\Omega_i \subset \mathbb{R}$, and let $\{V_{i,n}(\Omega_i)\}_{n\in\mathbb{Z}}$ form a nested sequence of subspaces of the function space $X(\Omega_i)$, with

$$V_{i,p}(\Omega_i) \subset V_{i,q}(\Omega_i) \subset X(\Omega_i) \quad \text{for} \quad p \leq q\,, \tag{2.31}$$

and let each sequence of (one-dimensional) operators $\{R_{i,n}\}_{n\in\mathbb{Z}}$ form an NSR. If we take for the corresponding reconstructions $\{P_{i,n}\}_{n\in\mathbb{Z}}$ the natural injection, then the tensor-product restrictions $\{R_{\boldsymbol{n}}\}$ and prolongations $\{P_{\boldsymbol{n}}\}$ form a CSR and a CSP respectively. (So, together they form a *commutative set of transfer operators.*)

Proof: By the nesting of the subspaces it is immediate that for each i the set $\{P_{i,n}\}_{n\in\mathbb{Z}}$ forms an NSP (Section 2.1). Further, using the fact that the prolongation is the natural injection, we can identify restrictions R_i with the corresponding projections Π_i, and by Lemma 2.7 we see that for each i we have $\Pi_{i,p}\Pi_{i,q} = \Pi_{i,q}\Pi_{i,p} = \Pi_{i,p}$ if $p \leq q$. Thus, for each i it follows that the sequences $\{P_{i,n}\}_{n\in\mathbb{Z}}$ and $\{R_{i,n}\}_{n\in\mathbb{Z}}$ are CSP and SCR. Now we prove that $\{R_{\boldsymbol{n}}\}$ is an CSR by showing $R_{\boldsymbol{m}}R_{\boldsymbol{n}} = R_{\min(\boldsymbol{m},\boldsymbol{n})}$ as follows.

Let $\boldsymbol{l} = \min(\boldsymbol{m},\boldsymbol{n})$ and let $v \in X(\Omega)$ be arbitrary, then we can write

$$v(\boldsymbol{x}) = \sum_{\boldsymbol{a}\in A} c_{\boldsymbol{a}} u_{\boldsymbol{a}}(\boldsymbol{x})$$

and

$$R_{\boldsymbol{m}} v(\boldsymbol{x}) = \sum_{\boldsymbol{a}} c_{\boldsymbol{a}} R_{\boldsymbol{m}} u_{\boldsymbol{a}}(\boldsymbol{x}) = \sum_{\boldsymbol{a}} c_{\boldsymbol{a}} \prod_i R_{i,m_i} u_{i,a_i}(x_i)\,.$$

Similarly,

$$\begin{aligned}
R_{\boldsymbol{n}}R_{\boldsymbol{m}}v(\boldsymbol{x}) &= \textstyle\sum_{\boldsymbol{a}} c_{\boldsymbol{a}}\ R_{\boldsymbol{n}} \prod_i R_{i,m_i} u_{i,a_i}(x_i)\\
&= \textstyle\sum_{\boldsymbol{a}} c_{\boldsymbol{a}} \prod_i R_{i,n_i} R_{i,m_i} u_{i,a_i}(x_i)\\
&= \textstyle\sum_{\boldsymbol{a}} c_{\boldsymbol{a}} \prod_i R_{i,\min(m_i,n_i)} u_{i,a_i}(x_i)\\
&= \textstyle\sum_{\boldsymbol{a}} c_{\boldsymbol{a}}\ R_{\min(\boldsymbol{m},\boldsymbol{n})} \prod_i u_{i,a_i}(x_i)\\
&= R_{\min(\boldsymbol{m},\boldsymbol{n})}\ \textstyle\sum c_{\boldsymbol{a}} \prod_i u_{i,a_i}(x_i)\\
&= R_{\min(\boldsymbol{m},\boldsymbol{n})}\ v(\boldsymbol{x})\,.
\end{aligned}$$

Hence $R_{\boldsymbol{n}}R_{\boldsymbol{m}} = R_{\min(\boldsymbol{m},\boldsymbol{n})}$, and, thus, $\Pi_{\boldsymbol{n}}\Pi_{\boldsymbol{m}} = \Pi_{\min(\boldsymbol{m},\boldsymbol{n})}$. Now Theorem (2.14) shows that $\{R_{\boldsymbol{n}}\}$ is an CSR and $\{P_{\boldsymbol{n}}\}$ is an NSP.

To prove that $\{P_{\boldsymbol{n}}\}$ is an CSP, we have to show $V_{\min(\boldsymbol{m},\boldsymbol{n})} = V_{\boldsymbol{m}} \cap V_{\boldsymbol{n}}$, or

$$\otimes_i V_{i,\min(m_i,n_i)} = \otimes_i V_{i,m_i} \cap \otimes_i V_{i,n_i}\,. \tag{2.32}$$

As for each i we know that $\{V_{i,n}(\Omega_i)\}_{n\in\mathbb{Z}}$ is a nested sequence of subspaces of $X_i(\Omega_i)$, we can construct a sequentially ordered set of basis functions $B_i = \{u_{i,b}\}_b$ in $X_i(\Omega_i)$, such that $n_i < m_i$ implies $u_{i,b} \in V_{i,n_i} \Rightarrow u_{i,b} \in V_{i,m_i}$. It follows that we have $V_{i,m_i} = \mathrm{Span}(\{u_{i,l} \in B_i | u_{i,l} \in V_{i,m_i}\})$ and similarly

$$V_{i,\min(n_i,m_i)} = \mathrm{Span}(\{u_{i,l} \in B_i | u_{i,l} \in V_{i,n_i} \cap V_{i,m_i}\})\,.$$

So we see

$$\begin{aligned}
&\operatorname*{Span}_{u_{i,l_i}\in B_i \cap V_{i,\min(m_i,n_i)}} \textstyle\prod_i u_{i,l_i} = \\
&\operatorname*{Span}_{u_{i,m_i}\in B_i \cap V_{i,m_i}} \textstyle\prod_i u_{i,m_i} \bigcap \operatorname*{Span}_{u_{i,n_i}\in B_i \cap V_{i,n_i}} \textstyle\prod_i u_{i,n_i}\,,
\end{aligned}$$

which is equivalent with (2.32). □

EXAMPLE 2.24 *Piecewise constant approximation.*

If we consider $L_2^{\mathrm{loc}}(\Omega) = \overline{X(\Omega)}$ and we choose for $R_{i,n}$ the one-dimensional L_2-projection $R_{i,n}$:

$X(\Omega_i) \to V_n(\Omega_i) \subset X(\Omega_i)$, where $\Omega_i \subset \mathbb{R}$ and $V_n(\Omega_i)$ is the space of piecewise constant functions on dyadic intervals (i.e., if $R_{i,n}$ denotes: taking mean values over intervals $[j2^{-n}, (j+1)2^{-n}]$ in the i-th coordinate direction), then, for each i, the restrictions $\{R_{i,n}\}_n$ form a one-dimensional NSR. The corresponding reconstructions $P_{i,n}$ represent piecewise constant interpolation over dyadic intervals. This makes the prolongations $\{P_{i,n}\}_n$ an NSP.

Then, as a consequence of the above theorem, $\{R_{\boldsymbol{k}}\}$ and $\{P_{\boldsymbol{k}}\}$ are commutative transfer operators, i.e., $\{R_{\boldsymbol{k}}\}$ is a CSR and $\{P_{\boldsymbol{k}}\}$ a CSP.

EXAMPLE 2.25 *Piecewise linear approximation.*
If we select the restriction $R_{i,n} : C^0(\Omega_i) \to V_n(\Omega_i) \subset \mathbb{R}^{\mathbb{Z}}$ to be: taking function values at dyadic points $j2^{-n}$ in the interval $\Omega_i \subset \mathbb{R}$, then $\{R_{i,n}\}_n$ is an NSR. Corresponding reconstructions $P_{i,n}$, defined by piecewise linear interpolation over dyadic intervals, make the prolongations $\{P_{i,n}\}_n$ an NSP.

As a consequence of the previous theorem, with $C^0(\Omega) = \overline{X(\Omega)}$, the tensor product operators $\{R_{\boldsymbol{k}}\}$ and $\{P_{\boldsymbol{k}}\}$, defined on $X(\Omega)$, are commutative (are a CSR and a CSP respectively). The restriction $R_{\boldsymbol{k}}$ takes the function values at grid points $\Omega_{\boldsymbol{n}}^+$, and the prolongation $P_{\boldsymbol{k}}$ makes a multi-linear interpolation over cells in $\Omega_{\boldsymbol{n}}$.

In the above examples, with $V_{\boldsymbol{n}} \subset X$ we took for the reconstruction $P_{\boldsymbol{n}}$ the natural injection (the identity in X). In this way we may identify $R_{\boldsymbol{n}}$ and $\Pi_{\boldsymbol{n}}$. It appears that in both cases, i.e. for the piecewise constant and the piecewise linear approximation, we have a projection $\Pi_{\boldsymbol{n}}$ of the form

$$\Pi_{\boldsymbol{n}} = \prod_{j=1}^{d} \Pi_{n_j \boldsymbol{e}_j} , = \prod_{j=1}^{d} R_{n_j \boldsymbol{e}_j} .$$

Here $R_{n_j \boldsymbol{e}_j} : X(\Omega) \to X(\Omega)$ is the operator on the tensor product space $X(\Omega)$ such that

$$R_{n_j \boldsymbol{e}_j} u_{\boldsymbol{a}}(\boldsymbol{x}) = R_{j,n_j} u_{j,a_j}(x_j) \cdot \prod_{i \neq j}^{d} u_{i,a_i}(x_i)$$

In the following section we consider the case of nested subspaces $\{V_{\boldsymbol{n}}\}$ with $V_{\boldsymbol{n}} \subset X$ and $X = \overline{\bigcup_{\boldsymbol{n}} V_{\boldsymbol{n}}}$, and where all spaces $V_{\boldsymbol{n}}$ are spanned by dilations of a single function $\phi(\boldsymbol{x})$, together with all its dyadic translations. This leads to the more-dimensional *multiresolution analysis* or *MRA*. In this case the spaces $W_{\boldsymbol{m}} = \mathrm{Ran}(Q_{\boldsymbol{m}})$ correspond with more-dimensional wavelet spaces.

2.6. More-dimensional MRA and wavelets

It will be convenient if

(i) we can make an arbitrarily accurate approximation of any function $u \in X$ by taking the multi-integer $\boldsymbol{n}$ large enough. Moreover, it will be convenient

(ii) all spaces $\{\mathrm{Ran}(P_{\boldsymbol{n}})\}$ or $\{V_{\boldsymbol{n}}\}$ have a similar structure, and

(iii) there is a clear relation between the spaces in $\{\mathrm{Ran}(P_{\boldsymbol{n}})\}$ or $\{V_{\boldsymbol{n}}\}$.

In order to create such a structure, in this section we introduce the multidimensional multiresolution analysis. For this purpose we will restrict ourselves to Hilbert spaces. First we introduce the important notion of frame.

DEFINITION 2.26 A sequence $\{x_n\}$ in a Hilbert space H is a *frame* if there exist numbers $A, B > 0$ such that for all $x \in H$ we have

$$A\|x\|^2 \leq \sum_n |(x, x_n)|^2 \leq B\|x\|^2. \tag{2.33}$$

The numbers A, B are called *frame bounds* . The frame is *tight* if $A = B$. The frame is *exact* if it ceases to be a frame whenever any single element is deleted from the sequence. If the sequence $\{x_n\}$ satisfies (only) the second part of the inequality (2.33) then the sequence is called a *Bessel sequence.*

Having introduced the exact frame, we can define the partially ordered, more-dimensional multiresolution analysis. Notice that this is different from the more-dimensional multiresolution analysis introduced in [1], which considers a sequentially ordered nested set of approximating spaces.

DEFINITION 2.27 Let $\Omega = \mathbb{R}^d$ and let $X(\Omega)$ be a Hilbert space of functions defined on Ω. A *multidimensional multiresolution analysis* of $X(\Omega)$, is a **partially ordered** set of **closed** linear subspaces

$$\{V_{\boldsymbol{n}} | V_{\boldsymbol{n}} \subset X(\Omega)\}_{\boldsymbol{n}\in\mathbb{Z}^d}$$

with the four properties:

$$\bigcap_{\boldsymbol{n}} V_{\boldsymbol{n}} = \{0\}; \qquad \overline{\bigcup_{\boldsymbol{n}} V_{\boldsymbol{n}}} = X(\Omega), \tag{2.34a}$$

$$f(\boldsymbol{x})\in V_{\boldsymbol{n}} \Leftrightarrow f(2^{\boldsymbol{m}}\boldsymbol{x})\in V_{\boldsymbol{n}+\boldsymbol{m}} \quad \forall \boldsymbol{n}, \boldsymbol{m}\in\mathbb{Z}^d, \tag{2.34b}$$

$$f(\boldsymbol{x})\in V_{\boldsymbol{n}} \Leftrightarrow f(\boldsymbol{x} - 2^{-\boldsymbol{n}}\boldsymbol{k})\in V_{\boldsymbol{n}} \quad \forall \boldsymbol{n}, \boldsymbol{k}\in\mathbb{Z}^d, \tag{2.34c}$$

$$\exists \phi\in V_{\boldsymbol{0}} : \ \{\phi(\boldsymbol{x} - \boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d} \text{ is an exact frame for } V_{\boldsymbol{0}}. \tag{2.34d}$$

The function $\phi(\boldsymbol{x})$ in (2.34d) is called the *father function* or the *scaling function* of the multiresolution analysis.

For $\Omega = \mathbb{R}^d$ the tensor product Examples 2.24 and 2.25 in Section 2.5 also yield examples of a multidimensional MRA.

For piecewise constant interpolation we take $X(\Omega) = L^2(\mathbb{R}^d)$ as the starting point. The characteristic function on the unit cube (the more-dimensional Haar function) is the scaling function ϕ. The set $\{V_{\boldsymbol{n}}\}$ contains the spaces of piecewise constant functions on $\Omega_{\boldsymbol{n}}$, and a CSR is obtained by $R_{\boldsymbol{n}} : X(\Omega) \to V_{\boldsymbol{n}}$, the L^2-projection. It is obvious that in this case the set $\{\phi(\boldsymbol{x} - \boldsymbol{k})\}_{\boldsymbol{k}}$ is an orthonormal basis and hence an exact frame with bounds $A = B = 1$.

For piecewise linear interpolation we take $X(\Omega) = H^{\boldsymbol{e}}(\mathbb{R}^d)$ as the Hilbert space. The set $\{V_{\boldsymbol{n}}\}$ contains the space of piecewise d-linear functions, determined by their nodal values at $\Omega^+_{\boldsymbol{n}}$. A CSR is obtained by $R_{\boldsymbol{n}} : X(\Omega) \to V_{\boldsymbol{n}}$, the piecewise d-linear interpolation at $\Omega^+_{\boldsymbol{n}}$. Here, the d-linear finite-elemen basis function is the scaling function ϕ. By Theorem ??? from [10] it is easily seen that in this case $\{\phi(\boldsymbol{x} - \boldsymbol{k})\}_{\boldsymbol{k}}$ is an exact frame, with as frame bounds the extreme eigenvalues of the frame operator $S : L^2(\Omega) \to L^2(\Omega)$ defined by $Su = \sum_{\boldsymbol{n}}(u, \phi(\boldsymbol{x} - \boldsymbol{k}))\phi(\boldsymbol{x} - \boldsymbol{k})$. Bounds for these extreme eigenvalues are $A = 3^{-d}$ and $B = 1$ respectively.

As in the tensor product case, we take for the reconstruction the natural injection $P_{\boldsymbol{n}} : V_{\boldsymbol{n}} \to X$ so that $R_{\boldsymbol{n}} \equiv \Pi_{\boldsymbol{n}}$ for all $\boldsymbol{n}\in\mathbb{Z}^d$.

More-dimensional wavelets

A *wavelet space* $W_{\boldsymbol{n}} \subset V_{\boldsymbol{n}}$, a closed subspace of $V_{\boldsymbol{n}}$ which contains those functions in $V_{\boldsymbol{n}}$ that cannot be represented in any of the function spaces on the next coarser level, i.e. these functions are in $V_{\boldsymbol{n}}$ but not in Span$(V_{\boldsymbol{n}-\boldsymbol{e}_1}, \cdots, V_{\boldsymbol{n}-\boldsymbol{e}_d})$. Thus $W_{\boldsymbol{n}} \subset V_{\boldsymbol{n}}$ is a closed subspace so that

$$V_{\boldsymbol{n}} = W_{\boldsymbol{n}} \ \oplus \text{Span}(V_{\boldsymbol{n}-\boldsymbol{e}_1}, \cdots, V_{\boldsymbol{n}-\boldsymbol{e}_d}), \tag{2.35}$$

This means that $W_{\boldsymbol{n}}$ contains the 'difference information' that is available in the fine grid $V_{\boldsymbol{n}}$ but not in the span of the coarser grids $V_{\boldsymbol{n}-\boldsymbol{e}_1}$, $V_{\boldsymbol{n}-\boldsymbol{e}_2}$, $\cdots$ and $V_{\boldsymbol{n}-\boldsymbol{e}_d}$.

The space $W_{\boldsymbol{n}}$ is the *complement* of $\mathrm{Span}(V_{\boldsymbol{n}-\boldsymbol{e}_1}, \cdots, V_{\boldsymbol{n}-\boldsymbol{e}_d})$ in $V_{\boldsymbol{n}}$. Of course, this complement is not uniquely determined. If we want we can make use of the Hilbert space structure and consider the (unique) *orthogonal complement*

$$W_{\boldsymbol{n}} \perp \mathrm{Span}(V_{\boldsymbol{n}-\boldsymbol{e}_1}, \cdots, V_{\boldsymbol{n}-\boldsymbol{e}_d}). \tag{2.36}$$

This choice corresponds with $R_{\boldsymbol{n}} : X \to V_{\boldsymbol{n}}$ being the orthogonal projection. However, in many cases we will use spaces $W_{\boldsymbol{n}}$ that don't satisfy this orthogonality property!

As soon as we have selected a CSR $\{R_{\boldsymbol{n}}\}$, then corresponding pre-wavelet spaces are defined as in Section 2.4. These pre-wavelet spaces on an MRA are wavelet spaces.

In the case of an MRA no coarsest grid exists, so that (2.28) gives

$$V_{\boldsymbol{n}} = \bigoplus_{\boldsymbol{j} \leq \boldsymbol{n}} W_{\boldsymbol{j}}. \tag{2.37}$$

Because of property (2.34.a) we can decompose the space $X(\Omega)$ in

$$X(\Omega) = \bigoplus_{\boldsymbol{j} \in \mathbb{Z}^d} W_{\boldsymbol{j}} \tag{2.38}$$

so that we can write any $u \in X(\Omega)$ as $u = \sum_{\boldsymbol{j} \in \mathbb{Z}^d} w_{\boldsymbol{j}}$ with $w_{\boldsymbol{j}} \in W_{\boldsymbol{j}}$. A restriction $R_{\boldsymbol{n}} : X(\Omega) \to V_{\boldsymbol{n}}$ is now determined by

$$R_{\boldsymbol{n}} \left(\sum_{\boldsymbol{j}} w_{\boldsymbol{j}} \right) = \sum_{\boldsymbol{j} \leq \boldsymbol{n}} w_{\boldsymbol{j}}. \tag{2.39}$$

By Definition 2.19 we recognise the direct hierarchical surplus

$$Q_{\boldsymbol{n}} : X(\Omega) \to \mathrm{Ran}(Q_{\boldsymbol{n}}) = W_{\boldsymbol{n}}. \tag{2.40}$$

We see that there is no coarsest grid and we can decompose $R_{\boldsymbol{n}}$ as

$$R_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \leq \boldsymbol{n}} Q_{\boldsymbol{k}}. \tag{2.41}$$

The four relations (2.34.a) to (2.34.d) imply that also the spaces $W_{\boldsymbol{n}}$ are scaled versions of one space $W_{\boldsymbol{0}}$,

$$f(\boldsymbol{x}) \in W_{\boldsymbol{n}} \Leftrightarrow f(2^{-\boldsymbol{n}}\boldsymbol{x}) \in W_{\boldsymbol{0}}, \quad \forall \boldsymbol{n} \in \mathbb{Z}^d, \tag{2.42}$$

and, moreover, that they are translation invariant for the discrete translations $2^{-\boldsymbol{n}}\mathbb{Z}^d$,

$$f(\boldsymbol{x}) \in W_{\boldsymbol{0}} \Leftrightarrow f(\boldsymbol{x}-\boldsymbol{k}) \in W_{\boldsymbol{0}}. \quad \forall \boldsymbol{n} \in \mathbb{Z}^d. \tag{2.43}$$

As soon as we find a function $\psi(\boldsymbol{x})$ with the property that $\psi(\boldsymbol{x}-\boldsymbol{k})$, $\boldsymbol{k} \in \mathbb{Z}^d$, is a basis of $W_{\boldsymbol{e}}$, then by a simple rescaling we see that $\psi(2^{\boldsymbol{n}}\boldsymbol{x}-\boldsymbol{k})$, yields a basis of $W_{\boldsymbol{n}+\boldsymbol{e}}$. Such a function is the more-dimensional generalisation of a *wavelet* [3]. Because of (2.38) the full collection

$$\left\{ \psi_{\boldsymbol{n},\boldsymbol{k}}(\boldsymbol{x}) \mid \psi_{\boldsymbol{n},\boldsymbol{k}}(\boldsymbol{x}) = \psi(2^{\boldsymbol{n}}\boldsymbol{x}-\boldsymbol{k}),\ \boldsymbol{n},\boldsymbol{k} \in \mathbb{Z}^d \right\}$$

is a basis of $X(\mathbb{R}^d)$.

3. Piecewise approximation in d dimensions

3.1. *Piecewise constant approximation*

We approximate $u \in X(\Omega) = L^p_{\mathrm{loc}}(\Omega)$ by $u_{\boldsymbol{n}} \in V_{\boldsymbol{n}}$, in the space of piecewise constant functions on $\Omega_{\boldsymbol{n}}$, i.e. in

$$V_{\boldsymbol{n}} = \mathrm{Span}(\{\phi_{\boldsymbol{n}\boldsymbol{j}}\}) , \tag{3.1}$$

with, for some $q \geq 1$ or $q = \infty$,

$$\begin{array}{l} \phi_{\boldsymbol{n}\boldsymbol{j}}(\boldsymbol{x}) = 2^{|\boldsymbol{n}|/q} \phi(2^{\boldsymbol{n}} \boldsymbol{x} - \boldsymbol{j}) , \\ \phi(\boldsymbol{x}) = \prod_{j=1}^{d} \chi_{[0,1]}(x_j) , \\ \text{with } \chi_{[0,1]}(x) \text{ the } \textit{characteristic function} \text{ on the unit interval.} \end{array} \tag{3.2}$$

This clearly describes a basis for a tensor product space, and we may write

$$V_{\boldsymbol{n}} = V_{\boldsymbol{n}}(\Omega) = \bigotimes_{j=1}^{d} V_{n_j}(\Omega_j) , \tag{3.3}$$

the tensor product of spaces $V_{n_j}(\Omega_j)$. These V_{n_j} are the spaces of piecewise constant functions with meshwidth $h_j = 2^{-n_j}$ on $\Omega_j \subset \mathbb{R}$. The corresponding grid of cells on the Cartesian product of $\{\Omega_j\}$, is denoted by $\Omega_{\boldsymbol{n}}$. The cell centers are denoted by by $\Omega^*_{\boldsymbol{n}}$.

We define the restriction $R_{\boldsymbol{n}}$ as the projection

$$\begin{array}{llll} R_{\boldsymbol{n}}: & X & \to & V_{\boldsymbol{n}} \subset X \\ & u & \mapsto & u_{\boldsymbol{n}} = R_{\boldsymbol{n}} u \, , \text{with} \\ & u_{\boldsymbol{n},\boldsymbol{i}} & = & u_{\boldsymbol{n}}((\boldsymbol{i} + \boldsymbol{e}/2)\boldsymbol{h}) = 2^{+|\boldsymbol{n}|} \int_{\Omega_{\boldsymbol{n}\boldsymbol{i}}} u(\xi) \, d\Omega \, . \end{array} \tag{3.4}$$

This restriction is of type (2.30), and $R_{\boldsymbol{n}} = R_{n_1,\cdots,n_d}$ can be decomposed as

$$R_{\boldsymbol{n}} = \prod_{j=1}^{d} R_{n_j \boldsymbol{e}_j} , \tag{3.5}$$

where $R_{n_j \boldsymbol{e}_j} u(\boldsymbol{x})$ is the function, piecewise constant in the j-th coordinate direction on a partitioning $\Omega_{\boldsymbol{n}}$, so that

$$R_{n_j \boldsymbol{e}_j} u(\boldsymbol{x}) = 2^{n_j} \int_{\boldsymbol{x}-(h/2)\boldsymbol{e}_j}^{\boldsymbol{x}+(h/2)\boldsymbol{e}_j} u(\xi_1, \cdots, \xi_d) \, d\xi_j$$

for all $\boldsymbol{x}$ with $(x_j 2^{n_j} \pm 1/2) \in \mathbb{Z}$.

In the special case $X = L^2(\Omega)$, the space $X = X(\Omega)$ is a Hilbert space, and $\{\phi_{\boldsymbol{n}\boldsymbol{j}}\}$ is an orthogonal (orthonormal if $q = 2$) basis in $V_{\boldsymbol{n}}$. In this case $R_{\boldsymbol{n}}$ is the orthogonal projection $L^2(\Omega) \to V_{\boldsymbol{n}}$. For $\Omega = \mathbb{R}^d$, the set $\{V_{\boldsymbol{n}}\}$ as defined in (3.1)-(3.2) is a typical MRA. This is no longer the case if we consider a bounded domain Ω, but the decomposition as treated in Section 2 still can be used in the case of a bounded domain.

It is easily checked that the more-dimensional wavelet $\psi(\boldsymbol{x}) \in W_{\boldsymbol{e}}$, corresponding with the piecewise constant scaling function $\phi(\boldsymbol{x}) \in V_{\boldsymbol{0}}$, from the previous section, is the more-dimensional *elementary checkerboard function* given by

$$\psi(\boldsymbol{x}) = \left\{ \begin{array}{ccl} 0 & \text{if} & \boldsymbol{x} \notin \Omega_{\boldsymbol{0},\boldsymbol{0}} \, , \\ (-1)^{|\boldsymbol{k}|} & \text{if} & \boldsymbol{x} \in \Omega_{\boldsymbol{0},\boldsymbol{0}} \text{ and } \boldsymbol{x} \in \Omega_{\boldsymbol{e},\boldsymbol{k}} \, . \end{array} \right. \tag{3.6}$$

This function is the tensor product of the Haar-wavelet.

In wavelet theory the spaces $W_{\boldsymbol{n}}$ are labeled *channels*, and the distinct channels are linearly independent. The first decomposition of an arbitrary function from $X(\Omega)$ consists in writing $u(\boldsymbol{x}) = \sum_{\boldsymbol{n}} w_{\boldsymbol{n}}(\boldsymbol{x})$, where $w_{\boldsymbol{n}} \in W_{\boldsymbol{n}}$ with $\boldsymbol{n} \in \mathbb{Z}^d$, according to (2.38).

Each subspace $W_{\boldsymbol{n}}$, has its natural basis, the *standard basis* ,

$$\left\{ \psi_{\boldsymbol{n},\boldsymbol{k}}(\boldsymbol{x}) \mid \psi_{\boldsymbol{n},\boldsymbol{k}}(\boldsymbol{x}) = 2^{|\boldsymbol{n}-\boldsymbol{e}|/q} \psi(2^{\boldsymbol{n}-\boldsymbol{e}}\boldsymbol{x} - \boldsymbol{k}),\ \boldsymbol{k} \in \mathbb{Z}^d \right\}, \tag{3.7}$$

of functions with a minimal support. We see that $\psi = \psi_{\boldsymbol{e},\boldsymbol{o}} \in V_{\boldsymbol{e}}$ is a function with the unit cube $\Omega_{\boldsymbol{o},\boldsymbol{o}}$ as support. The basis function $\psi_{\boldsymbol{n},\boldsymbol{k}}$ is a scaled, elementary checkerboard function, that may be characterised either by its support, which is a single cell in $\Omega_{\boldsymbol{n}-\boldsymbol{e}}$, or by the centerpoint of this cell, $z_{\boldsymbol{n}-\boldsymbol{e},\boldsymbol{k}} = 2^{-|\boldsymbol{n}-\boldsymbol{e}|}(\boldsymbol{k} + \boldsymbol{e}/2)$.

On the open unit cube $\Omega = (0,1)^d$ we consider the $2^{|\boldsymbol{n}|}$-dimensional spaces $V_{\boldsymbol{n}} = \otimes_{j=1}^d V_{n_j}$, the tensor product of $V_{n_j}((0,1))$, the spaces of piecewise constant functions with meshwidth $h_j = 2^{-n_j}$ in the j-th coordinate direction. For functions defined on $\Omega = (0,1)^d$ we can write relation (2.38) as

$$X(\Omega) = \bigoplus_{\boldsymbol{n} \geq \boldsymbol{o}} W_{\boldsymbol{n}}, \tag{3.8}$$

and make a decomposition in channels correspondingly. Each subspace $W_{\boldsymbol{n}+\boldsymbol{e}}$, now with $\boldsymbol{n} \geq \boldsymbol{o}$, has its standard basis $\psi_{\boldsymbol{n}+\boldsymbol{e},\boldsymbol{k}}$:

$$\left\{ \psi_{\boldsymbol{n}+\boldsymbol{e},\boldsymbol{k}}(\boldsymbol{x}) \mid \psi_{\boldsymbol{n}+\boldsymbol{e},\boldsymbol{k}}(\boldsymbol{x}) = 2^{|\boldsymbol{n}|/q} \psi(2^{\boldsymbol{n}}\boldsymbol{x} - \boldsymbol{k}),\ \boldsymbol{o} \leq \boldsymbol{k} < 2^{\boldsymbol{n}} \right\}. \tag{3.9}$$

For $\Omega = (0,1)^d$, the exceptions related with the boundary are found in the spaces $W_{\boldsymbol{n}}$ with a zero index (i.e. $\|\boldsymbol{n}\| = 0$). These $W_{\boldsymbol{n}}$ have basis functions with different shapes. They are derived from the corresponding functions for the unbounded case, but their support is restricted to Ω. Their corresponding nodal points $z_{\boldsymbol{n}-\boldsymbol{e},\boldsymbol{k}}$, are found on the boundary $\partial\Omega = \overline{\Omega} \backslash \Omega$ instead of in the interior. For $\|\boldsymbol{n}\| = 0$ we have $W_{\boldsymbol{n}}$ spanned by a basis

$$\left\{ \psi_{\boldsymbol{n},\boldsymbol{k}}(\boldsymbol{x}) \;\middle|\; \begin{array}{ll} 0 \leq k_j < 2^{n_j - 1} & \text{if } \; n_j \neq 0 \\ k_j = 0 & \text{if } \; n_j = 0 \\ & j = 1, \cdots, d \end{array} \right\}. \tag{3.10}$$

Taking such modifications into account, both for $\Omega = (0,1)^d$ and for $\Omega = \mathbb{R}^d$, for each $u \in L^p_{\text{loc}}(\Omega)$ we may write a *hierarchical expansion* (a wavelet expansion) according to (2.38) or (3.8), as

$$u(\boldsymbol{x}) = \sum_{\boldsymbol{n} \in \mathbb{Z}} w_{\boldsymbol{n}} = \sum_{\boldsymbol{n},\boldsymbol{k}} c_{\boldsymbol{n}\boldsymbol{k}} \psi_{\boldsymbol{n},\boldsymbol{k}}(\boldsymbol{x}) = \sum_{\boldsymbol{n},\boldsymbol{k}} c_{\boldsymbol{n}\boldsymbol{k}} \; \widehat{\psi}(2^{\boldsymbol{n}}\boldsymbol{x} - \boldsymbol{k}), \tag{3.11}$$

where $\widehat{\psi}$ is simply

$$\widehat{\psi}(\boldsymbol{x}) = \left\{ \begin{array}{ccll} = & \text{sign}\,\|\boldsymbol{x}\| & \text{if} & \max(x_1, ..., x_d) < 1, \\ = & 0 & \text{if} & \max(x_1, ..., x_d) > 1, \end{array} \right.$$

and $c_{\boldsymbol{n}\boldsymbol{k}} = 0$ for all $\boldsymbol{k}$ with $\|\boldsymbol{k}\|$ even.

3.2. Piecewise linear approximation

We approximate $u \in X = C^0(\Omega)$ by $u_{\boldsymbol{n}} \in V_{\boldsymbol{n}}$, in the space of piecewise d-linear functions on $\Omega_{\boldsymbol{n}}$, i.e. in

$$V_{\boldsymbol{n}} = \text{Span}(\{\phi_{\boldsymbol{n}\boldsymbol{j}}\}), \tag{3.12}$$

with, for some $q \geq 1$ or $q = \infty$,

$$\begin{array}{l} \phi_{\boldsymbol{nj}}(\boldsymbol{x}) = 2^{|\boldsymbol{n}|/q} \phi(2^{\boldsymbol{n}} \boldsymbol{x} - \boldsymbol{j}) \,, \\ \phi(\boldsymbol{x}) = \prod_{j=1}^{d} \Lambda(x_j) \,, \\ \text{with } \Lambda(x) = \max(0, 1 - |x|) \text{ the usual } \textit{hat function.} \end{array} \tag{3.13}$$

Clearly this is a basis for a tensor product space as (3.3), where V_{n_j} are spaces of piecewise linear functions on a partitioning of Ω_j with meshwidth $h_j = 2^{-n_j}$. The set of nodal points $\{\boldsymbol{j}2^{\boldsymbol{n}}\}_{\boldsymbol{j}\in\mathbb{Z}^d}$ in Ω is denoted by $\Omega^+_{\boldsymbol{n}}$.
Here we define the restriction $R_{\boldsymbol{n}}$ as the projection

$$\begin{array}{rlcl} R_{\boldsymbol{n}}: & X & \rightarrow & V_{\boldsymbol{n}} \subset X \\ & u & \mapsto & u_{\boldsymbol{n}} = R_{\boldsymbol{n}} u \,, \\ & u_{\boldsymbol{n}}(\boldsymbol{x}) & = & u(\boldsymbol{x}) \, \forall \boldsymbol{x} \in \Omega^+_{\boldsymbol{n}} \,. \end{array} \tag{3.14}$$

The restriction is also of type (2.30) and the operator $R_{\boldsymbol{n}} = R_{n_1,\cdots,n_d}$ can be decomposed as (3.5) where $R_{n_j \boldsymbol{e}_j} u(\boldsymbol{x})$ is a function, piecewise linear in the j-th coordinate direction, on a partitioning $\Omega_{\boldsymbol{n}}$, such that $R_{n_j \boldsymbol{e}_j} u(\boldsymbol{x}) = u(\boldsymbol{x})$ for all $\boldsymbol{x}$ with $x_j/h_j \in \mathbb{Z}$.

It is clear that there exists a basis-function in $V_{\boldsymbol{n}}$ for each nodal point $\boldsymbol{x}_{\boldsymbol{nk}}$ in $\overline{\Omega}^+_{\boldsymbol{n}}$. If and only iff $\|\boldsymbol{k}\|$ is even, there exists a 'parent grid' $\overline{\Omega}^+_{\boldsymbol{m}}$ with $\boldsymbol{m} \leq \boldsymbol{n}$ and $\boldsymbol{m} \neq \boldsymbol{n}$, for which $x_{\boldsymbol{nk}} \in \overline{\Omega}^+_{\boldsymbol{m}}$. Hence, in this case each wavelet space $W_{\boldsymbol{n}}$ has its natural basis

$$\left\{ \phi_{\boldsymbol{nk}} \mid \boldsymbol{k} \in \mathbb{Z}^d \text{ with } \|\boldsymbol{k}\| \text{ odd } \right\} \tag{3.15}$$

On the closed cube $\overline{\Omega} = [0,1]^d$ we consider the $\prod_{j=1}^{d} (2^{n_j} + 1)$-dimensional spaces $V_{\boldsymbol{n}} = V_{\boldsymbol{n}}(\Omega) = \bigotimes_{j=1}^{d} V_{n_j}([0,1])$. With *homogeneous Dirichlet boundary conditions*, the dimension of the corresponding space $V^0_{\boldsymbol{n}} \subset V_{\boldsymbol{n}}$ is $\prod_{j=1}^{d} (2^{n_j} - 1)$. It is immediately clear that typical FE-basis functions for $V_{\boldsymbol{n}}$ are the d-dimensional hat-functions: functions that vanish on all but one point of $\overline{\Omega}^+_{\boldsymbol{n}}$. Each such FE basis function of $V^0_{\boldsymbol{n}}$ is characterised by an interior point from $\Omega^+_{\boldsymbol{n}}$.

We notice that for $\overline{\Omega} = [0,1]^d$ we have

$$V_{\boldsymbol{n}} = \{\mathbf{o}\} \text{ except for } \boldsymbol{n} \geq \mathbf{o} \,, \tag{3.16}$$

and

$$V^0_{\boldsymbol{n}} = \{\mathbf{o}\} \text{ except for } \boldsymbol{n} \geq \boldsymbol{e} \,. \tag{3.17}$$

With $W_{\boldsymbol{n}}$ (or $W^0_{\boldsymbol{n}}$) we denote the subspace of $V_{\boldsymbol{n}}$ (respectively $V^0_{\boldsymbol{n}}$) of functions that vanish at the gridpoints of all $\overline{\Omega}^+_{\boldsymbol{n}-\boldsymbol{e}_j}$ (respectively $\Omega^+_{\boldsymbol{n}-\boldsymbol{e}_j}$), $j = 1, \cdots, d$. From (3.12) we see that for $\Omega = \mathbb{R}^d$

$$W_{\boldsymbol{n}} = \mathrm{Span}(\{\phi_{\boldsymbol{nj}} \mid \|\boldsymbol{j}\| \text{ odd } \}) \,. \tag{3.18}$$

and for $\Omega = [0,1]^d$ we see that $\mathbf{o} \leq \boldsymbol{j} \leq 2^{\boldsymbol{n}}$ and

$$W_{\boldsymbol{n}} = \mathrm{Span}(\left\{ \phi_{\boldsymbol{nj}} \;\middle|\; \begin{array}{ll} j_i \text{ odd}, \; 0 < j_i < 2^{n_i} & \text{if } n_i > 0, \\ j_i = 0, 1 & \text{if } n_i = 0, \\ & i = 1, \ldots, d \end{array} \right\}) \,. \tag{3.19}$$

Clearly $W_{\boldsymbol{n}} = W^0_{\boldsymbol{n}} = \{0\}$, except for $\boldsymbol{n} \geq \mathbf{o}$. If $\|\boldsymbol{n}\| = 0$ we see that $V^0_{\boldsymbol{n}} = \{0\}$ and $W_{\boldsymbol{n}}$ is spanned by FE basis functions that are characterised by boundary points on the unit cube. Thus, the trace of a function on the boundary is exclusively approximated by elements of $W_{\boldsymbol{n}}$ with $\|\boldsymbol{n}\| = 0$. Further we see $W^0_{\boldsymbol{n}} = W_{\boldsymbol{n}}$ if $\boldsymbol{n} \geq \boldsymbol{e}$. Apparently

$$V_{\boldsymbol{n}} = \bigoplus_{\mathbf{o} \leq \boldsymbol{k} \leq \boldsymbol{n}} W_{\boldsymbol{k}}\,, \tag{3.20}$$

and

$$V^0_{\boldsymbol{n}} = \bigoplus_{\boldsymbol{e} \leq \boldsymbol{k} \leq \boldsymbol{n}} W_{\boldsymbol{k}}\,. \tag{3.21}$$

4. Error estimates for regular grids

The decompositions of type (2.38) allow the approximation of a sufficiently smooth function in $X(\Omega)$ by a series with elements in $W_{\boldsymbol{j}}$. To obtain an impression of the quality of these expansions we derive some error estimates.

4.1. Estimates for piecewise constant approximation

As the case where domain boundaries are present, is the more general one, we study the case $\Omega = (0,1)^d$. To quantify the error of approximation on Ω, we introduce for $u \in C^{\boldsymbol{e}}(\overline{\Omega})$ the seminorm

$$|u| = \max_{\boldsymbol{x} \in \Omega} \left| D^{\boldsymbol{e}} u(\boldsymbol{x}) \right| + \max_{\substack{\mathbf{o} \leq \boldsymbol{p} \leq \boldsymbol{e} \\ 0 < |\boldsymbol{p}| < d}} \max_{\boldsymbol{x} \in \overline{\Omega} \setminus \Omega} \left| D^{\boldsymbol{p}} u(\boldsymbol{x}) \right| . \tag{4.1}$$

Now we derive the following

Theorem 4.1 If we consider an expansion of a $C^{\boldsymbol{e}}(\overline{\Omega})$-function, u, in piecewise constant functions on the grid $\Omega_{\boldsymbol{n}}$, for an arbitrary $\boldsymbol{n} \in \mathbb{Z}^d$, $\boldsymbol{n} > \mathbf{o}$, and if we write

$$R_{\boldsymbol{n}} u = \sum_{\mathbf{o} \leq \boldsymbol{m} \leq \boldsymbol{n}} w_{\boldsymbol{m}}\,, \tag{4.2}$$

with $w_{\boldsymbol{m}} \in W_{\boldsymbol{m}}$, $\mathbf{o} \leq \boldsymbol{m} \leq \boldsymbol{n}$, then, for $\boldsymbol{m} \neq \mathbf{o}$ we have

$$\|w_{\boldsymbol{m}}\|_{L_2(\Omega)} \leq 2^{-d/2}\, 2^{-|\boldsymbol{m}|}\, |u|\,, \tag{4.3}$$

and an estimate for the approximation error

$$\|u - R_{\boldsymbol{n}} u\|_{L_2(\Omega)} \leq (2/3)^{d/2}\, \|\boldsymbol{h}_{\boldsymbol{n}}\|\, |u|\,. \tag{4.4}$$

Proof: We take $\{\psi_{\boldsymbol{mk}}\}$ as a basis in $W_{\boldsymbol{m}}$, $\boldsymbol{e} \leq \boldsymbol{m} \leq \boldsymbol{n}$. All these functions form an $L_2(\Omega)$-orthonormal set (orthonormal Haar basis) and they are orthogonal to all functions in $W_{\boldsymbol{n}}$, $\boldsymbol{n} \neq \boldsymbol{m}$.

Thus, we find (4.2) with $w_{\boldsymbol{m}} = \sum_{\boldsymbol{k}} a_{\boldsymbol{mk}}\, \psi_{\boldsymbol{mk}}$, where

$$a_{\boldsymbol{mk}} = (u, \psi_{\boldsymbol{m},\boldsymbol{k}}) = \int_{\Omega} u\, \psi_{\boldsymbol{m},\boldsymbol{k}}\, d\Omega = \int_{\Omega_{\boldsymbol{m}-\boldsymbol{e},\boldsymbol{k}}} u\, \psi_{\boldsymbol{m},\boldsymbol{k}}\, d\Omega\,.$$

For $\boldsymbol{m} \geq \boldsymbol{e}$ the point $\boldsymbol{z}_{\boldsymbol{m}-\boldsymbol{e},\boldsymbol{k}}$ lies in the interior of Ω and the estimate holds with

$$|u| = \max_{\boldsymbol{x}} \left| D^{\boldsymbol{e}} u(\boldsymbol{x}) \right| .$$

Viz., by Taylor expansion around $\boldsymbol{z}_{\boldsymbol{m}-\boldsymbol{e},\boldsymbol{k}}$, we have

$$\begin{aligned}
|a_{\boldsymbol{m}\boldsymbol{k}}| &= \left|\int_{\Omega_{\boldsymbol{m}-\boldsymbol{e},\boldsymbol{k}}} u\,\psi_{\boldsymbol{m},\boldsymbol{k}}\; d\Omega\right| \\
&\leq \left|\int_{\Omega_{\boldsymbol{m}-\boldsymbol{e},\boldsymbol{k}}} \|\!|\boldsymbol{x}-\boldsymbol{z}_{\boldsymbol{m}-\boldsymbol{e},\boldsymbol{k}}\|\!|\; |u| \;\; \psi_{\boldsymbol{m},\boldsymbol{k}}\; d\Omega\right| \\
&= \int_{\Omega_{\boldsymbol{m}-\boldsymbol{e},\boldsymbol{k}}} |\, \|\!|\boldsymbol{x}-\boldsymbol{z}_{\boldsymbol{m}-\boldsymbol{e},\boldsymbol{k}}\|\!|\,|\; |u| \;\; 2^{|\boldsymbol{m}-\boldsymbol{e}|/2}\; d\Omega \\
&= |u|\; 2^{|\boldsymbol{m}-\boldsymbol{e}|/2} \int_{\Omega_{\boldsymbol{m}-\boldsymbol{e},\boldsymbol{k}}} |\, \|\!|\boldsymbol{x}-\boldsymbol{z}_{\boldsymbol{m}-\boldsymbol{e},\boldsymbol{k}}\|\!|\,|\; d\Omega \\
&= |u|\; 2^{|\boldsymbol{m}-\boldsymbol{e}|/2} 2^d \prod_{j=1}^d \int_0^{2^{-m_j}} \xi_j\; d\xi_j \\
&= |u|\; 2^{|\boldsymbol{m}-\boldsymbol{e}|/2} 2^d \prod_{j=1}^d \left(\tfrac{1}{2} 2^{-2m_j}\right) \\
&= |u|\; 2^{|\boldsymbol{m}-\boldsymbol{e}|/2} 2^{-2|\boldsymbol{m}|} = \; |u|\; 2^{|\boldsymbol{m}-\boldsymbol{e}|/2} 2^{-2|\boldsymbol{m}|} \\
&= |u|\; 2^{-d/2} 2^{-3|\boldsymbol{m}|/2}
\end{aligned} \tag{4.5}$$

For $\boldsymbol{m} \neq \mathbf{o}$, $\boldsymbol{m} \not\geq \boldsymbol{e}$, i.e. for $\psi_{\boldsymbol{m}\boldsymbol{k}}$ with an $\boldsymbol{m}$-component equal to zero, the point $\boldsymbol{z}_{\boldsymbol{m}-\boldsymbol{e},\boldsymbol{k}}$ lies on the boundary and the function $\psi_{\boldsymbol{m},\boldsymbol{k}}$ is constant in one direction over the whole domain Ω, and it is of Haar-wavelet type for the non-zero indices (or index). In this case the orthonormal basis functions are $2^{\beta/2}\psi_{\boldsymbol{m},\boldsymbol{k}}$, where β is the number of zeroes in $\boldsymbol{m}$. Nevertheless, provided that $\boldsymbol{m} \neq \mathbf{o}$, in this situation the same estimate (4.5) holds with, e.g. if $m_1 = 0$ and $m_j \geq 1$ for $j = 2, ..., d$,

$$|u| = \max_{\boldsymbol{x}} \left| \frac{\partial^{d-1} u(\boldsymbol{x})}{\partial x_2 \cdots \partial x_d} \right| .$$

Hence, the estimate (4.5) holds for $\boldsymbol{m} \geq \mathbf{o}$, $\boldsymbol{m} \neq \mathbf{o}$, if we use the seminorm (4.1), and we find

$$\|w_{\boldsymbol{m}}\|^2 = \sum_{\boldsymbol{k}} |a_{\boldsymbol{m}\boldsymbol{k}}|^2 \leq \sum_{\boldsymbol{k}} 2^{-d}\, 2^{-3|\boldsymbol{m}|} |u|^2 = 2^{-d}\, 2^{-2|\boldsymbol{m}|} |u|^2 . \tag{4.6}$$

so that $\|w_{\boldsymbol{m}}\| \leq 2^{-d/2} 2^{-|\boldsymbol{m}|} |u|$, which leads to (4.3) and (4.4) because

$$\begin{aligned}
\|u - R_{\boldsymbol{n}} u\|^2 &= \sum_{\substack{m_1 > n_1 \\ \text{or } \cdots \text{ or} \\ m_d > n_d}} \|w_{\boldsymbol{m}}\|^2 \leq \sum_{j=1}^d \sum_{\substack{\boldsymbol{m} \geq \mathbf{o} \\ m_j > n_j}} 2^{-d}\, 2^{-2|\boldsymbol{m}|}\; |u|^2 \\
&\leq 2^{-d}\; |u|^2 \sum_{j=1}^d \sum_{\substack{\boldsymbol{m} \geq \mathbf{o} \\ m_j > n_j}} (1/4)^{m_1 + \cdots + m_d} \\
&\leq 2^{-d}\; |u|^2\; (4/3)^d \sum_{j=1}^d (1/4)^{n_j} \leq |u|^2\; (2/3)^d \sum_{j=1}^d h_{n_j}^2 \\
&\leq |u|^2\; (2/3)^d\; \|\boldsymbol{h}_{\boldsymbol{n}}\|^2 .
\end{aligned}$$

□

If we have no further a-priori knowledge about u, the most efficient approximation will be one with $h_1 = \cdots = h_d$ because this equalises the main terms in the error bound. We see that the truncation error for $u - R_{\boldsymbol{n}} u$ is not particularly promising or surprising: the major part of the error is produced by the largest meshwidth: $(\max(h_1, \cdots, h_d))^{d/2}$, whereas the total number of degrees of freedom for an element in $V_{\boldsymbol{n}}$ is $2^{|\boldsymbol{n}|}$.

4.2. Estimates for piecewise linear approximation

For a function $u \in C^{\mathbf{o}}(\Omega)$ we consider piecewise linear approximation as in Section 3.2. We approximate u by $u_{\boldsymbol{n}} \in V_{\boldsymbol{n}}$, where $V_{\boldsymbol{n}}$ is the space of piecewise d-linear functions on $\Omega_{\boldsymbol{n}}$. We take $u_{\boldsymbol{n}}$ such that $u_{\boldsymbol{n}}(\boldsymbol{x}) = u(\boldsymbol{x})$ for all $x \in \Omega^{+}_{\boldsymbol{n}}$ and we write

$$u_{\boldsymbol{n}}(\boldsymbol{x}) = \sum_{\boldsymbol{j}} d_{\boldsymbol{nj}} \phi_{\boldsymbol{nj}}(\boldsymbol{x}) \,, \tag{4.7}$$

where $\phi_{\boldsymbol{nj}}(\boldsymbol{x})$ is defined by (3.13).

With $u_{\boldsymbol{n}} \in V_{\boldsymbol{n}}$ the piecewise linear approximation on $\Omega_{\boldsymbol{n}}$ of the function $u \in C^{\mathbf{o}}(\Omega)$, we make the hierarchical decomposition $V_{\boldsymbol{n}} = \bigoplus_{\boldsymbol{k} \leq \boldsymbol{n}} W_{\boldsymbol{k}}$, and write

$$u_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \leq \boldsymbol{n}} w_{\boldsymbol{k}} \,, \quad w_{\boldsymbol{k}} \in W_{\boldsymbol{k}} \,, \tag{4.8}$$

where

$$w_{\boldsymbol{k}}(\boldsymbol{x}) = \sum_{\boldsymbol{j}} c_{\boldsymbol{kj}} \phi_{\boldsymbol{kj}}(\boldsymbol{x}) \,, \tag{4.9}$$

with $c_{\boldsymbol{kj}} = 0$ for all $\boldsymbol{j}$ with $\|\boldsymbol{j}\|$ even.

In practice the coefficients $c_{\boldsymbol{kj}}$, $\|\boldsymbol{j}\|$ odd, are computed as *hierarchical surplus coefficients*, by taking the difference between the value $u(\boldsymbol{j}\boldsymbol{h}_{\boldsymbol{k}})$ and the interpolant from coarser grids. This is most conveniently formulated by introducing stencil notation. Therefore, we introduce the *difference operator*

$$\nabla_{\boldsymbol{h}} u(\boldsymbol{z}) = u(\boldsymbol{z} + \boldsymbol{h}) - u(\boldsymbol{z}) \,, \tag{4.10}$$

and the usual central difference approximation for the second derivative by *stencil* notation, as

$$\left[\frac{1}{2}, -1, \frac{1}{2}\right]_{h_j \boldsymbol{e}_j} u(\boldsymbol{z}) = \frac{1}{2} \nabla^2_{h_j \boldsymbol{e}_j} u(\boldsymbol{z} - h_j \boldsymbol{e}_j) \,.$$

With this notation we write an expression for the hierarchical coefficients in a piecewise linear approximation. We see that d-linear interpolation leads to the following expression for the hierarchical surplus coefficient

$$c_{\boldsymbol{kj}} \, \|\boldsymbol{h}_{\boldsymbol{k}}\|^{-\frac{1}{q}} = \prod_{j=1,d} \left[-\frac{1}{2}, 1, -\frac{1}{2}\right]_{h_j \boldsymbol{e}_j} u(\boldsymbol{j}\boldsymbol{h}_{\boldsymbol{k}}) \,. \tag{4.11}$$

Notice that the factor $\|\boldsymbol{h}_{\boldsymbol{k}}\|^{\frac{1}{q}}$ cancels the scaling factor $2^{|\boldsymbol{k}|/q}$ in the definition of $\phi_{\boldsymbol{kj}}$, so that the function $u(\boldsymbol{x})$ is expanded as

$$u(\boldsymbol{x}) \approx \sum_{\boldsymbol{k},\boldsymbol{j}} c_{\boldsymbol{kj}} \, \|\boldsymbol{h}_{\boldsymbol{k}}\|^{-\frac{1}{q}} \, \phi(2^{\boldsymbol{k}} \boldsymbol{x} - \boldsymbol{j}) \,.$$

An expression for the coefficient $c_{\boldsymbol{kj}}$ is found in the following lemma.

LEMMA 4.2

Let $u \in C^{\boldsymbol{e}+\boldsymbol{m}}$, for a given $\boldsymbol{m}$ with $\mathbf{o} \leq \boldsymbol{m} \leq \boldsymbol{e}$, and let)

$$L_{\boldsymbol{nj}}(\boldsymbol{x}) = 2^{-|\boldsymbol{n}|} \phi(2^{\boldsymbol{n}} \boldsymbol{x} - \boldsymbol{j}) \,, \tag{4.12}$$

then, for each $\phi_{\boldsymbol{nj}} \in W_{\boldsymbol{n}}$, $\|\boldsymbol{n}\| \neq 0$, $\|\boldsymbol{j}\|$ odd, we have

$$\begin{aligned} \|\boldsymbol{h}_{\boldsymbol{n}}\|^{-\frac{1}{q}} \, |c_{\boldsymbol{nj}}| &= \textstyle\prod_{i=1}^{d} \left[-\frac{1}{2}, 1, -\frac{1}{2}\right]_{h_i \boldsymbol{e}_i} u(\boldsymbol{j}\boldsymbol{h}_{\boldsymbol{n}}) \\ &= (-1)^{|\boldsymbol{e}+\boldsymbol{m}|} \, 2^{-d} \int_{\Omega} D^{\boldsymbol{e}+\boldsymbol{m}} u(\boldsymbol{x}) \, D^{\boldsymbol{e}-\boldsymbol{m}} L_{\boldsymbol{nj}}(\boldsymbol{x}) \, d\Omega \,. \end{aligned}$$

Proof: We see that for $\|\boldsymbol{n}\| \neq 0$, $\|\boldsymbol{j}\|$ odd, each $L_{\boldsymbol{nj}}$ has a support in the interior of Ω. Taking this into account, we give the proof after a coordinate translation with $2^{\boldsymbol{m}}\boldsymbol{j}$ then we see that for all i, $0 \le i \le d$,

$$\begin{array}{rl} & \sum_{s_i=-1,1} \int_0^{s_i h_i} D^{\boldsymbol{e}_i} u(\boldsymbol{z})\, dz_i \\ = & \sum_{s_i=-1,1} u(\boldsymbol{z}) \Big|_{z_i=0}^{s_i h_i} \\ = & \sum_{s_i=-1,1} \nabla_{s_i h_i \boldsymbol{e}_i} u(\boldsymbol{z}) \Big|_{z_i=0} \\ = & [1,-2,1]_{h_i \boldsymbol{e}_i} u(\boldsymbol{z}) \Big|_{z_i=0} \end{array}$$

and hence

$$\begin{array}{rl} & \prod_{i=1,d} \left[-\frac{1}{2}, 1, -\frac{1}{2}\right]_{h_i \boldsymbol{e}_i} u(\mathbf{o}) \\ = & (\frac{-1}{2})^d \prod_{i=1,d} \left[\sum_{s_i=-1,1} \int_0^{s_i h_i} D^{\boldsymbol{e}_i} \bullet\, dx_i\right] u(x) \\ = & (\frac{-1}{2})^d \sum_{s_1,\cdots,s_d=-1,1} \int_0^{s_1 h_1} \cdots \int_0^{s_d h_d} D^{\boldsymbol{e}} u(\boldsymbol{x})\; dx_1 \cdots dx_d \\ = & (\frac{-1}{2})^d \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} D^{\boldsymbol{e}} u(\boldsymbol{x}) \prod_{i=1}^d \text{sign } (x_i) \;\; dx_1 \cdots dx_d \\ = & (\frac{-1}{2})^d \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} D^{\boldsymbol{e}} u(\boldsymbol{x}) \prod_{i=1}^d \left(2^{-n_i} \frac{d}{dx_i} \Lambda(2^{n_i} x_i)\right) dx_1 \cdots dx_d \\ = & (\frac{-1}{2})^d \int D^{\boldsymbol{e}} u(\boldsymbol{x}) \prod_{i=1}^d \left(2^{-n_i} \frac{d}{dx_i} \Lambda(2^{n_i} x_i)\right) d\Omega \\ = & (\frac{-1}{2})^d \int D^{\boldsymbol{e}} u(\boldsymbol{x})\; 2^{-|\boldsymbol{n}|}\; D^{\boldsymbol{e}} \phi(2^{|\boldsymbol{n}|} \boldsymbol{x})\, d\Omega \\ = & (\frac{-1}{2})^d \int D^{\boldsymbol{e}} u(\boldsymbol{x})\; D^{\boldsymbol{e}} L_{\boldsymbol{n}\mathbf{o}}(\boldsymbol{x})\, d\Omega \\ = & (-1)^{|\boldsymbol{e}+\boldsymbol{m}|}\, 2^{-d} \int D^{\boldsymbol{e}+\boldsymbol{m}} u(\boldsymbol{x})\, D^{\boldsymbol{e}-\boldsymbol{m}} L_{\boldsymbol{n}\mathbf{o}}(\boldsymbol{x})\, d\Omega \end{array}$$

□

Remarks:

- For $\|\boldsymbol{n}\| = 0$, (i.e. for boundary points), the same formula holds, provided that the formula is restricted to the lower dimensional boundary manifold (e.g. the face or the edge of the unit cube).
- For an $\boldsymbol{m}$ with $\mathbf{o} \le \boldsymbol{m} \le \boldsymbol{e}$ we derive an expression for $\|D^{\boldsymbol{m}}\phi\|_p$, with ϕ given by (3.13) as follows

  $$\begin{array}{rl} \|D^{\boldsymbol{m}}\phi\|_p^p & = \int_\Omega \prod_i |D^{m_i} \Lambda(x_i)|^p d\Omega \\ & = \prod_i \left(\int_{-1}^1 |\, D^{m_i}(1-|x_i|)\,|^p\, dx_i\right) \\ & = 2^d\,(p+1)^{-|\boldsymbol{e}-\boldsymbol{m}|} \end{array}$$

  So that

  $$\|D^{\boldsymbol{m}}\phi\|_p = 2^{d/p}(p+1)^{-|\boldsymbol{e}-\boldsymbol{m}|/p} \tag{4.13}$$

- In (3.13) we have $\phi_{\boldsymbol{nj}} = 2^{|\boldsymbol{n}|/q}\phi(2^{\boldsymbol{n}}\boldsymbol{x} - \boldsymbol{j})$, and hence

  $$\begin{array}{rl} \|D^{\boldsymbol{m}}\phi_{\boldsymbol{nj}}\|_p^p & = \int 2^{|\boldsymbol{n}|p/q}\, |D^{\boldsymbol{m}}\phi(2^{\boldsymbol{n}}\boldsymbol{x} - \boldsymbol{j})|^p\, d\Omega \\ & = 2^{|\boldsymbol{n}|p/q} \;\; \int |D_x^{\boldsymbol{m}}\phi(2^{\boldsymbol{n}}\boldsymbol{x})|^p\, d(2^{|\boldsymbol{n}|}\boldsymbol{x}) \;\; 2^{-|\boldsymbol{n}|} \\ & = 2^{|\boldsymbol{n}|p/q} \;\; \int |2^{\boldsymbol{m}\circ\boldsymbol{n}} D_z^{\boldsymbol{m}}\phi(\boldsymbol{z})|^p\, d\boldsymbol{z} \;\; 2^{-|\boldsymbol{n}|} \\ & = 2^{|\boldsymbol{n}|(p/q-1)}\, 2^{|\boldsymbol{m}\circ\boldsymbol{n}|p}\, \|D^{\boldsymbol{m}}\phi\|_p^p\,, \end{array}$$

  Here, $2^{\boldsymbol{m}\circ\boldsymbol{n}} = \prod_{i=1}^d 2^{m_i n_i} = \prod_{i=1}^d h_{n_i}^{-m_i} = \boldsymbol{h}_{\boldsymbol{n}}^{-\boldsymbol{m}}$, so that for arbitrary $\boldsymbol{j}$,

  $$\|D^{\boldsymbol{m}}\phi_{\boldsymbol{nj}}\|_p = \|\boldsymbol{h}_{\boldsymbol{n}}\|^{(1/p-1/q)}\, \boldsymbol{h}_{\boldsymbol{n}}^{-\boldsymbol{m}}\, \|D^{\boldsymbol{m}}\phi\|_p\,. \tag{4.14}$$

  This means that the norm $\|\phi_{\boldsymbol{nj}}\|_p$ is independent of the level $\boldsymbol{n}$ iff we take $q = p$.

- To obtain error estimates in the approximation Theorem 4.3 we compute an expression for $\|D^{\boldsymbol{m}} L_{\boldsymbol{nj}}(\boldsymbol{x})\|_p$, with $L_{\boldsymbol{nj}}$ as introduced in (4.12), in particular for $p = 1, 2, \infty$.

$$\begin{aligned} \left\| D^{\boldsymbol{m}} L_{\boldsymbol{nj}}(\boldsymbol{x}) \right\|_p^p &= \|\boldsymbol{h_n}\|^{p+1}\, \boldsymbol{h_n}^{-p\boldsymbol{m}}\, \|D^{\boldsymbol{m}}\phi\|_p^p \\ &= \|\boldsymbol{h_n}\|^{p+1}\, \boldsymbol{h_n}^{-p\boldsymbol{m}}\, 2^d\, (p+1)^{-|\boldsymbol{e}-\boldsymbol{m}|} \end{aligned} \quad .$$

So that we may conclude, also considering the special case $p = \infty$,

$$\begin{array}{lcrll} \||D^{\boldsymbol{m}} L_{\boldsymbol{nj}}(\boldsymbol{x})\|_1 &=& & \|\boldsymbol{h_n}\|^2 & (\boldsymbol{h_n}/2)^{-\boldsymbol{m}}\,, \\ \||D^{\boldsymbol{m}} L_{\boldsymbol{nj}}(\boldsymbol{x})\|_2 &=& (2/3)^{d/2} & \|\boldsymbol{h_n}\|^{3/2} & (\boldsymbol{h_n}/\sqrt{3})^{-\boldsymbol{m}}\,, \\ \||D^{\boldsymbol{m}} L_{\boldsymbol{nj}}(\boldsymbol{x})\|_\infty &=& & \|\boldsymbol{h_n}\| & (\boldsymbol{h_n})^{-\boldsymbol{m}}\,. \end{array}$$

Using the above expressions and Lemma 4.2, we can derive the error estimates in the following theorem.

THEOREM 4.3 Let $u \in C_0^{\boldsymbol{e}+\boldsymbol{m}}(\Omega)$ be given for some $\boldsymbol{m}$ with $\boldsymbol{o} \leq \boldsymbol{m} \leq \boldsymbol{e}$, and let $u_{\boldsymbol{n}} \in V_{\boldsymbol{n}}$ be the piecewise linear approximation on $\Omega_{\boldsymbol{n}}$ of u, such that $u_{\boldsymbol{n}}(\boldsymbol{x}) = u(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega_{\boldsymbol{n}}^+$. If we make the hierarchical decomposition $V_{\boldsymbol{n}} = \oplus_{\boldsymbol{k} \leq \boldsymbol{n}} W_{\boldsymbol{k}}$, and write

$$u_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \leq \boldsymbol{n}} w_{\boldsymbol{k}}\,, \quad w_{\boldsymbol{k}} \in W_{\boldsymbol{k}}\,,$$

then we have the estimates

$$\begin{array}{lcll} \|w_{\boldsymbol{k}}\|_2 &\leq& \|D^{\boldsymbol{e}+\boldsymbol{m}}u\|_2 & \|\boldsymbol{h_k}\|^2\, 2^{-d/2}\, 3^{-|\boldsymbol{m}|/2}\, \boldsymbol{h_k}^{-(\boldsymbol{e}-\boldsymbol{m})}, \\ \|w_{\boldsymbol{k}}\|_\infty &\leq& \|D^{\boldsymbol{e}+\boldsymbol{m}}u\|_\infty & \|\boldsymbol{h_k}\|^2\, (\boldsymbol{h_k}/2)^{-(\boldsymbol{e}-\boldsymbol{m})}\,, \\ \|u - u_{\boldsymbol{n}}\|_2 &\leq& \|D^{\boldsymbol{e}+\boldsymbol{m}}u\|_2 & 2^{-d}\, 3^{-3|\boldsymbol{m}|/2} \sum_{i=1}^d h_{n_i}^{(1+m_i)}\,, \\ \|u - u_{\boldsymbol{n}}\|_\infty &\leq& \|D^{\boldsymbol{e}+\boldsymbol{m}}u\|_\infty & 6^{-|\boldsymbol{m}|} \sum_{i=1}^d h_{n_i}^{(1+m_i)}\,. \end{array}$$

Proof: Using (4.11) and Lemma 4.2 we can obtain estimates for the hierarchical coefficients $c_{\boldsymbol{kj}}$. We fix $\boldsymbol{k}$ and we derive, writing $\boldsymbol{h} = \boldsymbol{h_k}$,

$$\begin{array}{rcll} \|\boldsymbol{h}\|^{-\frac{1}{q}}\, |c_{\boldsymbol{kj}}| = & 2^{-d} & \left| \int_\Omega D^{\boldsymbol{e}+\boldsymbol{m}}u(\boldsymbol{x})\, \chi_{\boldsymbol{kj}}(\boldsymbol{x}) \right. & \left. D^{\boldsymbol{e}-\boldsymbol{m}} L_{\boldsymbol{kj}}(\boldsymbol{x})\, d\boldsymbol{x} \right| \\ \leq & 2^{-d} & \|D^{\boldsymbol{e}+\boldsymbol{m}}u\, \chi_{\boldsymbol{kj}}\|_\infty & \|D^{\boldsymbol{e}-\boldsymbol{m}} L_{\boldsymbol{kj}}\|_1 \\ \leq & 2^{-d} & \|D^{\boldsymbol{e}+\boldsymbol{m}}u\, \chi_{\boldsymbol{kj}}\|_\infty & \|\boldsymbol{h}\|^2\, (\boldsymbol{h}/2)^{-(\boldsymbol{e}-\boldsymbol{m})}\,, \end{array}$$

where $\chi_{\boldsymbol{kj}}$ is the characteristic function for the support of $L_{\boldsymbol{kj}}(\boldsymbol{x})$, or similarly

$$\begin{array}{rcll} \|\boldsymbol{h}\|^{-\frac{1}{q}}\, |c_{\boldsymbol{kj}}| = & 2^{-d} & \left| \int_\Omega D^{\boldsymbol{e}+\boldsymbol{m}}u(\boldsymbol{x})\, \chi_{\boldsymbol{kj}} \right. & \left. D^{\boldsymbol{e}-\boldsymbol{m}} L_{\boldsymbol{kj}}(\boldsymbol{x})\, d\boldsymbol{x} \right| \\ \leq & 2^{-d} & \|D^{\boldsymbol{e}+\boldsymbol{m}}u\, \chi_{\boldsymbol{kj}}\|_2 & \|D^{\boldsymbol{e}-\boldsymbol{m}} L_{\boldsymbol{kj}}\|_2 \\ \leq & 6^{-d/2} & \|D^{\boldsymbol{e}+\boldsymbol{m}}u\, \chi_{\boldsymbol{kj}}\|_2 & \|\boldsymbol{h}\|^{3/2}\, (\boldsymbol{h}/\sqrt{3})^{-(\boldsymbol{e}-\boldsymbol{m})}\,. \end{array}$$

We write $w_{\boldsymbol{k}} = \sum_{\boldsymbol{j}} c_{\boldsymbol{kj}} \phi_{\boldsymbol{kj}}$ with $\|\boldsymbol{j}\|$ odd, and we know that these functions $\{\phi_{\boldsymbol{kj}}\}_{\boldsymbol{j}}$, for fixed $\boldsymbol{k}$ have disjoint supports. Hence, for the hierarchical contribution,

$$\begin{array}{rcll} \|w_{\boldsymbol{k}}\|_2^2 = & & \|\sum_{\boldsymbol{i}} c_{\boldsymbol{ki}} \phi_{\boldsymbol{ki}}\|_2^2 = \sum_{\boldsymbol{i},\boldsymbol{j}} c_{\boldsymbol{ki}} c_{\boldsymbol{kj}} \int \phi_{\boldsymbol{ki}} \phi_{\boldsymbol{kj}}\, d\Omega & \\ = & & \sum_{\boldsymbol{i}} c_{\boldsymbol{ki}}^2 \|\phi_{\boldsymbol{ki}}\|_2^2 & \\ \leq & \|\boldsymbol{h}\|^{2/q} & \|\phi_{\boldsymbol{ki}}\|_2^2\, \sum_{\boldsymbol{j}} \|D^{\boldsymbol{e}+\boldsymbol{m}} u \chi_{\boldsymbol{kj}}\|_2^2\, 6^{-d} \|\boldsymbol{h}\|^3 (\boldsymbol{h}/\sqrt{3})^{-2(\boldsymbol{e}-\boldsymbol{m})} & \\ \leq & \|\boldsymbol{h}\|^4 & 2^{-d}\, \|D^{\boldsymbol{e}+\boldsymbol{m}}u\|_2^2\;\; \boldsymbol{h}^{-2(\boldsymbol{e}-\boldsymbol{m})}\, 3^{-|\boldsymbol{m}|}\,. & \end{array}$$

For the other norm $\|w_{\boldsymbol{k}}\|_\infty$ we obtain similarly

$$\begin{aligned}\|w_{\boldsymbol{k}}\|_\infty = & \ \|\sum_{\boldsymbol{j}} c_{\boldsymbol{kj}}\phi_{\boldsymbol{kj}}\|_\infty \\ \leq & \ \max_{\boldsymbol{j}} |c_{\boldsymbol{kj}}| \ \|\phi_{\boldsymbol{kj}}\|_\infty \\ \leq & \ \|\boldsymbol{h}\|^{\frac{1}{q}} \ \max_{\boldsymbol{j}} 2^{-d}\|D^{\boldsymbol{e}+\boldsymbol{m}}u\,\chi_{\boldsymbol{kj}}\|_\infty \, \|\boldsymbol{h}\|^2\,(h/2)^{-(\boldsymbol{e}-\boldsymbol{m})} \ \|\boldsymbol{h}\|^{-\frac{1}{q}} \\ \leq & \ 2^{-d}\,\|D^{\boldsymbol{e}+\boldsymbol{m}}u\|_\infty\,\|\boldsymbol{h}\|^2\,(\boldsymbol{h}/2)^{-(\boldsymbol{e}-\boldsymbol{m})}\end{aligned}$$

For the error, for $p=2$ or $p=\infty$, we get

$$\begin{aligned}\|u-u_{\boldsymbol{n}}\|_p = & \ \|\sum_{\boldsymbol{k}} w_{\boldsymbol{k}} - \sum_{\boldsymbol{k}\leq\boldsymbol{n}} w_{\boldsymbol{k}}\|_p \\ \leq & \ \sum_{\boldsymbol{k}\not\leq\boldsymbol{n}} \|w_{\boldsymbol{k}}\|_p \\ \leq & \ C_p\,\|D^{\boldsymbol{e}+\boldsymbol{m}}u\|_p \ \sum_{\boldsymbol{k}\not\leq\boldsymbol{n}} \|\boldsymbol{h}_{\boldsymbol{k}}\|^2\,\boldsymbol{h}_{\boldsymbol{k}}^{-(\boldsymbol{e}-\boldsymbol{m})} \\ = & \ C_p\,\|D^{\boldsymbol{e}+\boldsymbol{m}}u\|_p \ \sum_{\boldsymbol{k}\not\leq\boldsymbol{n}} \boldsymbol{h}_{\boldsymbol{k}}^{\boldsymbol{e}+\boldsymbol{m}}\end{aligned}$$

with $C_2 = 2^{-d/2}3^{-|\boldsymbol{m}|/2}$ or $C_\infty = 2^{-d}2^{|\boldsymbol{e}-\boldsymbol{m}|}$. This yields the above mentioned estimates, by taking into account that

$$\begin{aligned}&\sum_{\boldsymbol{k}\not\leq\boldsymbol{n}} \boldsymbol{h}_{\boldsymbol{k}}^{\boldsymbol{e}+\boldsymbol{m}} = \sum_{\boldsymbol{k}} \boldsymbol{h}_{\boldsymbol{k}}^{\boldsymbol{e}+\boldsymbol{m}} - \sum_{\boldsymbol{k}\leq\boldsymbol{n}} \boldsymbol{h}_{\boldsymbol{k}}^{\boldsymbol{e}+\boldsymbol{m}} = \\ &= \sum_{0\leq k_1,\ldots,0\leq k_d} \prod_{i=1}^d h_{k_i}^{1+m_i} - \sum_{0\leq k_1\leq n_i,\ldots,0\leq k_d\leq n_d} \prod_{i=1}^d h_{k_i}^{1+m_i} \\ &= \prod_{i=1}^d \sum_{0\leq k_i} h_{k_i}^{1+m_i} - \prod_{i=1}^d \sum_{0\leq k_i\leq n_i} h_{k_i}^{1+m_i} = \\ &= \prod_{i=1}^d \sum_{0\leq k_i} (1/2)^{k_i(1+m_i)} - \prod_{i=1}^d \sum_{0\leq k_i\leq n_i} (1/2)^{k_i(1+m_i)} = \\ &= \prod_{i=1}^d \frac{1}{1-(1/2)^{1+m_i}} - \prod_{i=1}^d \frac{1-(1/2)^{(1+n_i)(1+m_i)}}{1-(1/2)^{1+m_i}} = \\ &= \prod_{i=1}^d \frac{1}{1-(1/2)^{1+m_i}} \ \left[1 - \prod_{i=1}^d \left(1-(1/2)^{(1+n_i)(1+m_i)}\right)\right] = \\ &\leq 2^d\,(2/3)^{|\boldsymbol{m}|} \ \left[1 - \prod_{i=1}^d \left(1-2^{-(1+n_i)(1+m_i)}\right)\right] = \\ &\leq 2^d\,(2/3)^{|\boldsymbol{m}|} \ \sum_{i=1}^d 2^{-(1+n_i)(1+m_i)} = 3^{-|\boldsymbol{m}|} \ \sum_{i=1}^d h_{n_i}^{(1+m_i)}\end{aligned}$$

For $\boldsymbol{m}=\boldsymbol{e}$ this simply reads $\sum_{\boldsymbol{k}\not\leq\boldsymbol{n}} \|\boldsymbol{h}_{\boldsymbol{k}}\|^2 \leq 3^{-d}\,\|\boldsymbol{h}_{\boldsymbol{n}}\|^2$. □

Corollary 4.4

As a direct corollary we find

$$\begin{aligned}\|u-u_{\boldsymbol{n}}\|_2 \ &\leq \ 54^{-d/2} \ \|D^{2\boldsymbol{e}}u\|_2 \ \|\boldsymbol{h}_{\boldsymbol{n}}\|^2, \\ \|u-u_{\boldsymbol{n}}\|_\infty \ &\leq \ 6^{-d} \ \|D^{2\boldsymbol{e}}u\|_\infty \ \|\boldsymbol{h}_{\boldsymbol{n}}\|^2,\end{aligned}$$

and, for $p=2$ or $p=\infty$, and $\mathbf{o}\leq\boldsymbol{m}\leq\boldsymbol{e}$,

$$\|w_{\boldsymbol{k}}\|_p \ \leq \ C \ \|D^{\boldsymbol{e}+\boldsymbol{m}}u\|_p \ \ \boldsymbol{h}_{\boldsymbol{k}}^{\boldsymbol{e}+\boldsymbol{m}}, \tag{4.15}$$

and

$$\begin{aligned}\|u-u_{\boldsymbol{n}}\|_p \ &\leq \ C \ \|D^{\boldsymbol{e}+\boldsymbol{m}}u\|_p \ \sum_{i=1}^d h_{n_i}^{1+m_i}, \\ \|u-u_{\boldsymbol{n}}\|_p \ &\leq \ C \ \|D^{2\boldsymbol{e}}u\|_p \ \|\boldsymbol{h}_{\boldsymbol{n}}\|^2.\end{aligned} \tag{4.16}$$

From (4.15) also follows a bound by a $W_p^{\boldsymbol{e},\ell}$-norm, We immediately see that, for $0\leq\ell\leq d$,

$$\|w_{\boldsymbol{k}}\|_p \leq C\,\|u\|_{W_p^{\boldsymbol{e},\ell}} \min_{|\boldsymbol{m}|=\ell} \boldsymbol{h}_{\boldsymbol{k}}^{\boldsymbol{e}+\boldsymbol{m}} \leq C\,\|u\|_{W_p^{\boldsymbol{e},\ell}}\|\boldsymbol{h}_{\boldsymbol{k}}\|^{1+\ell/d}. \tag{4.17}$$

Remark:
We gave the proof of Theorem 4.3 for functions that vanish at the boundary of Ω. Taking into account the remark following Lemma 4.2, it is clear that similar estimates (with different constants and with terms including derivatives of u that are restricted to the boundary planes) also hold for functions with non-homogeneous boundary conditions.

5. Error estimates for sparse grids

5.1. Estimates for piecewise constant approximation

THEOREM 5.1 Let $\widehat{R}_n u$ be the piecewise constant approximation of a function $u \in L_2(\Omega)$ on a sparse grid on level n:

$$\widehat{R}_n u = \sum_{|\boldsymbol{k}| \le n} w_{\boldsymbol{k}}\,, \quad w_{\boldsymbol{k}} \in W_{\boldsymbol{k}}\,, \tag{5.1}$$

then, with $\|\boldsymbol{h}\| = 2^{-n}$, the volume of the finest cells, we have the estimate

$$\|u - \widehat{R}_n u\|_{L_2(\Omega)} \le C\,|u|\,\|\boldsymbol{h}\|\,\log^{(d-1)/2}\|\boldsymbol{h}\|\,. \tag{5.2}$$

Proof: To prove the theorem for the $L_2(\Omega)$-norm, we use (4.6), and the orthogonality of the hierarchical basis functions, to obtain

$$\begin{array}{rcl} \|u - \widehat{R}_n u\|^2_{L_2(\Omega)} & \le & \sum_{|\boldsymbol{k}|>n} \|w_{\boldsymbol{k}}\|^2_{L_2(\Omega)} \\ & \le & \sum_{|\boldsymbol{k}|>n} 2^{-d}\, 4^{-|\boldsymbol{k}|} |u|^2 \\ & = & 2^{-d}|u|^2 \sum_{|\boldsymbol{k}|>n} 2^{-2|\boldsymbol{k}|} \\ & = & 2^{-d}|u|^2 \sum_{l>n} 2^{-2l} \begin{pmatrix} l+d-1 \\ d-1 \end{pmatrix} \end{array} \tag{5.3}$$

We know

$$\begin{array}{rl} & \sum_{l>n} 2^{-2l} \begin{pmatrix} l+d-1 \\ d-1 \end{pmatrix} \\ = & 2^{-2(n+1)} \begin{pmatrix} n+d \\ d-1 \end{pmatrix} F(1, 1+n+d; 2+n; 1/4) \\ = & G(n,d) \end{array} \tag{5.4}$$

Here F is the hypergeometric function. It follows that

$$G(n,d) \sim \frac{n^{d-1} 2^{-2n}}{3(d-1)!} \quad \text{for} \quad n \to \infty\,,$$

where the asymptotic value is reached soon for small values of d. Hence

$$\begin{array}{rcl} \|u - R_n u\|^2 & \le & 2^{-d}\,|u|^2 \sum_{l>n} 2^{-2l} \begin{pmatrix} l+d-1 \\ d-1 \end{pmatrix} \\ & \le & 2^{-d}\,|u|^2 \;\; G(n,d) \\ & \le & 2^{-d}\,|u|^2 \;\; C_{nd} \frac{n^{d-1}2^{-2n}}{3(d-1)!} \end{array} \tag{5.5}$$

where C_{nd} is a constant that tends to one for large n. So, we conclude that

$$\|u - R_n u\|_{L_2(\Omega)} \le C\;|u|\;n^{(d-1)/2}\,2^{-n}\,,$$

which is equivalent with (5.2). □

To guarantee a small error on a regular grid, in (4.4) all cell edges h_j need to be small, but in (5.2) for the sparse grid only the volume $|\!|\!|\boldsymbol{h}|\!|\!|$ has to be small. Further, in the two-dimensional case, the estimate (5.2) is of a similar order of accuracy as (4.4), except for a logarithmic small factor. However, the number of degrees of freedom for the approximation (5.2) is significantly less. Namely, in the unit cube, for $R_{\boldsymbol{n}}u$ the number of degrees of freedom is $2^{|\boldsymbol{n}|}$, whereas for $\widehat{R}_n u$ it is $\mathcal{O}(n^{d-1}2^n)$, viz. $2n2^n + 1$ in the 2D case, and in the 3D-case e.g. $(n^2+n+2)2^n - 1$. Because significantly less degrees of freedom are involved in the approximation $\widehat{R}_n u$ than in the approximation of $R_{(n,n,n)}u$, i.e. less coefficients $a_{\boldsymbol{j},\boldsymbol{k}}$ and less gridpoints $z_{\boldsymbol{j},\boldsymbol{k}}$, in analogy to [7], we call the approximation $\widehat{R}_n u$ the *sparse grid approximation* and

$$\Omega_n^\star = \left\{ z_{\boldsymbol{j},\boldsymbol{k}} \mid z_{\boldsymbol{j},\boldsymbol{k}} \in Z_{\boldsymbol{n}}, |\boldsymbol{n}| \le n \right\}$$

is the *sparse (box) grid* for this approximation on level n.

5.2. Estimates for piecewise linear approximation

For piecewise linear approximation we use a *sparse (vertex) grid.* $\Omega_n^+ = \bigcup_{|\boldsymbol{k}|\le n} \Omega_{\boldsymbol{l}}^+ \bigcap \Omega$. A sparse grid approximation is obtained by interpolation on this grid by means of the space spanned by all $W_{\boldsymbol{k}}$ with $|\boldsymbol{k}| \le n$.

THEOREM 5.2 Let $\widehat{R}_n u$ be the piecewise d-linear approximation of a function $u \in C_0^{\boldsymbol{e}+\boldsymbol{m}}(\Omega)$, with $\mathbf{o} \le \boldsymbol{m} \le \boldsymbol{e}$, on a sparse grid on level n:

$$\widehat{R}_n u = \sum_{|\boldsymbol{k}|\le n} w_{\boldsymbol{k}}\,, \quad w_{\boldsymbol{k}} \in W_{\boldsymbol{k}}\,, \tag{5.6}$$

then, with $|\!|\!|\boldsymbol{h}|\!|\!| = 2^{-n}$, the volume of the finest cells, we have for $p = 2, \infty$, with $\boldsymbol{m} = \boldsymbol{e}$ the estimates

$$\|u - \widehat{R}_n u\|_p \le C \|D^{2\boldsymbol{e}} u\|_p \, |\!|\!|\boldsymbol{h}|\!|\!|^2 \log^{d-1} |\!|\!|\boldsymbol{h}|\!|\!|^{-1}\,, \tag{5.7}$$

and with $|\boldsymbol{m}| < d$ the estimates

$$\|u - \widehat{R}_n u\|_p \le C \|D^{\boldsymbol{e}+\boldsymbol{m}} u\|_p \, |\!|\!|\boldsymbol{h}|\!|\!| \log^{d-1-|\boldsymbol{m}|} |\!|\!|\boldsymbol{h}|\!|\!|^{-1}\,, \tag{5.8}$$

and with $0 \le \ell \le d$

$$\|u - \widehat{R}_n u\|_p \le C \|u\|_{W_p^{\boldsymbol{e},\ell}} \, |\!|\!|\boldsymbol{h}|\!|\!|^{1+\ell/d} \log^{d-1} |\!|\!|\boldsymbol{h}|\!|\!|^{-1}\,. \tag{5.9}$$

Proof: Using the estimates for $\|w_{\boldsymbol{k}}\|_p$ from Theorem 4.3, we prove, more generally, for some $\boldsymbol{m}$ with $\mathbf{o} \le \boldsymbol{m} \le \boldsymbol{e}$, and for $p = 2$ or $p = \infty$,

$$\begin{array}{rcl} \|u - \widehat{R}_n u\|_p & \le & \sum_{|\boldsymbol{k}|>n} \|w_{\boldsymbol{k}}\|_p \\ & \le & \sum_{|\boldsymbol{k}|>n} C\,\|D^{\boldsymbol{e}+\boldsymbol{m}} u\|_p \, \boldsymbol{h}_{\boldsymbol{k}}^{\boldsymbol{e}+\boldsymbol{m}} \\ & = & C\,\|D^{\boldsymbol{e}+\boldsymbol{m}} u\|_p \sum_{|\boldsymbol{k}|>n} \boldsymbol{h}_{\boldsymbol{k}}^{\boldsymbol{e}+\boldsymbol{m}} \\ & = & C\,\|D^{\boldsymbol{e}+\boldsymbol{m}} u\|_p \sum_{|\boldsymbol{k}|>n} |\!|\!|\boldsymbol{h}_{\boldsymbol{k}}|\!|\!| \prod_{j=1}^d 2^{-k_j m_j} \\ & = & C\,\|D^{\boldsymbol{e}+\boldsymbol{m}} u\|_p \sum_{l>n} 2^{-l} \sum_{|\boldsymbol{k}|=l} \prod_{j=1}^d 2^{-k_j m_j} \\ & = & C\,\|D^{\boldsymbol{e}+\boldsymbol{m}} u\|_p \sum_{l>n} 2^{-l} \sum_{|\boldsymbol{k}|=l} 2^{-\boldsymbol{k}\circ\boldsymbol{m}} \\ & \le & C\,\|D^{\boldsymbol{e}+\boldsymbol{m}} u\|_p \, 2^{-n} \left(C_1\, 2^{-n} n^{|\boldsymbol{m}|-1} + C_2 n^{d-|\boldsymbol{m}|-1}\right), \end{array} \tag{5.10}$$

with $C_1 = 0$ if $|\boldsymbol{m}| = 0$, and $C_2 = 0$ if $|\boldsymbol{m}| = d$. Hence, for $\boldsymbol{m} \ne \boldsymbol{e}$ we have

$$\|u - \widehat{R}_n u\|_p \le C \|D^{\boldsymbol{e}+\boldsymbol{m}} u\|_p \, |\!|\!|\boldsymbol{h}|\!|\!| \;\log^{d-1-|\boldsymbol{m}|} |\!|\!|\boldsymbol{h}|\!|\!|^{-1}\,.$$

Moreover, (5.10) yields, for $\boldsymbol{m} = \boldsymbol{e}$,

$$\|u - \widehat{R}_n u\|_p \le C \|D^{2\boldsymbol{e}} u\|_p \, \|\boldsymbol{h}\|^2 \; \log^{d-1} \|\boldsymbol{h}\|^{-1} \, .$$

Further, using the estimate (4.17) we obtain, similar to the proof for Theorem 5.1,

$$\begin{aligned}
\|u - \widehat{R}_n u\|_p &\le \textstyle\sum_{|\boldsymbol{k}|>n} \|w_{\boldsymbol{k}}\|_p \sum_{|\boldsymbol{k}|>n} C \, \|u\|_{W_p^{\boldsymbol{e},\ell}} \|\boldsymbol{h}_{\boldsymbol{k}}\|^{1+\ell/d} \\
&= C \, \|u\|_{W_p^{\boldsymbol{e},\ell}} \textstyle\sum_{|\boldsymbol{k}|>n} 2^{-|\boldsymbol{k}|\,(1+\ell/d)} \\
&= C \, \|u\|_{W_p^{\boldsymbol{e},\ell}} \textstyle\sum_{l>n} \begin{pmatrix} l+d-1 \\ d-1 \end{pmatrix} 2^{-l(1+\ell/d)} \\
&= C \, \|u\|_{W_p^{\boldsymbol{e},\ell}} \begin{pmatrix} n+d \\ d-1 \end{pmatrix} F(1, 1+n+d; 2+n; 2^{-(1+\ell/d)} 2^{-n(1+\ell/d)} \\
&\le C \, \|u\|_{W_p^{\boldsymbol{e},\ell}} 2^{-n(1+\ell/d)} \; n^{d-1}/(d-1)! \\
&\le C \, \|u\|_{W_p^{\boldsymbol{e},\ell}} \|\boldsymbol{h}\|^{1+\ell/d} \log^{d-1} \|\boldsymbol{h}\|^{-1} \, .
\end{aligned} \tag{5.11}$$

□

THEOREM 5.3 Let $\widehat{R}_n u$ be the piecewise d-linear approximation of a function $u \in C_0^{\boldsymbol{e},1}(\Omega)$ on a sparse grid on level n, as in Theorem 5.2, then, with $\|\boldsymbol{h}\| = 2^{-n}$, the volume of the finest cells, we have, for $p = 2$, $p = \infty$, the estimates

$$\|u - \widehat{R}_n u\|_{W_p^1} \le C \; \|\boldsymbol{h}\| \; \log^{d-1} \|\boldsymbol{h}\|^{-1} \; \|u\|_{W_p^{\boldsymbol{e},1}} = \tag{5.12}$$

If, moreover, we know $u \in C^{2\boldsymbol{e}}$, then

$$\|u - \widehat{R}_n u\|_{W_p^1} \le C \|\boldsymbol{h}\| \, \|D^{2\boldsymbol{e}} u\|_p \, . \tag{5.13}$$

Proof: Let u be sufficiently differentiable and let $\mathbf{o} \le \boldsymbol{m} \le \boldsymbol{e}$ and $|\boldsymbol{m}| \ge 1$, then
Part 1:

$$\begin{aligned}
\|D^{\boldsymbol{m}} w_{\boldsymbol{k}}\|_p &= \|D^{\boldsymbol{m}} \textstyle\prod_{j=1}^d (R_{\boldsymbol{k}} - R_{\boldsymbol{k}-\boldsymbol{e}_j}) u\|_p \\
&\le C^d \, \|D^{\boldsymbol{m}} \textstyle\prod_{j=1}^d h_{k_j} D^{\boldsymbol{e}_j} u\|_p \\
&\le C^d \, \|\boldsymbol{h}_{\boldsymbol{k}}\| \, \|D^{\boldsymbol{m}} D^{\boldsymbol{e}} u\|_p
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
\|D^{\boldsymbol{m}}(u - \widehat{R}_n u)\|_p &\le \|\textstyle\sum_{|\boldsymbol{k}|>n} D^{\boldsymbol{m}} w_{\boldsymbol{k}}\|_p \\
&\le \textstyle\sum_{|\boldsymbol{k}|>n} \|D^{\boldsymbol{m}} w_{\boldsymbol{k}}\|_p \\
&\le \textstyle\sum_{|\boldsymbol{k}|>n} C^d \, \|\boldsymbol{h}_{\boldsymbol{k}}\| \; \|D^{\boldsymbol{m}+\boldsymbol{e}} u\|_p \\
&\le C^d \, \|D^{\boldsymbol{m}+\boldsymbol{e}} u\|_p \; \textstyle\sum_{|\boldsymbol{k}|>n} \|\boldsymbol{h}_{\boldsymbol{k}}\| \\
&\le C^d \, \|D^{\boldsymbol{m}+\boldsymbol{e}} u\|_p \; \textstyle\sum_{l>n} \begin{pmatrix} l+d-1 \\ d-1 \end{pmatrix} 2^{-l} \\
&\le C^d \, \|D^{\boldsymbol{m}+\boldsymbol{e}} u\|_p \; \frac{n^{d-1} 2^{-n}}{(d-1)!} \bar{G}(n,d) \\
&\le C \, \|D^{\boldsymbol{m}+\boldsymbol{e}} u\|_p \; \|\boldsymbol{h}\| \; \log^{d-1} \|\boldsymbol{h}\|
\end{aligned} \tag{5.15}$$

Part 2:

$$\begin{aligned}
\|D^{\boldsymbol{m}}(u - \widehat{R}_n u)\|_p &\le \|\textstyle\sum_{|\boldsymbol{k}|>n} D^{\boldsymbol{m}} w_{\boldsymbol{k}}\|_p \\
&\le \textstyle\sum_{|\boldsymbol{k}|>n} \|D^{\boldsymbol{m}} \prod_{j=1}^d (R_{\boldsymbol{k}} - R_{\boldsymbol{k}-\boldsymbol{e}_j}) u\|_p \\
&\le \textstyle\sum_{|\boldsymbol{k}|>n} C \, \|\boldsymbol{h}_{\boldsymbol{k}}\|^2 h_{\boldsymbol{k}}^{-\boldsymbol{m}} \|D^{2\boldsymbol{e}} u\|_p \\
&\le C \, \|D^{2\boldsymbol{e}} u\|_p \textstyle\sum_{|\boldsymbol{k}|>n} \|\boldsymbol{h}_{\boldsymbol{k}}\|^2 h_{\boldsymbol{k}}^{-\boldsymbol{m}} \\
&\le C \, \|D^{2\boldsymbol{e}} u\|_p \textstyle\sum_{l>n} \, 2^{-2l} \sum_{|\boldsymbol{k}|=l} h_{\boldsymbol{k}}^{-\boldsymbol{m}} \\
&\le C \, \|D^{2\boldsymbol{e}} u\|_p \textstyle\sum_{l>n} \, 2^{-2l} \sum_{|\boldsymbol{k}|=l} 2^{\boldsymbol{k} \circ \boldsymbol{m}} \\
&= C \, \|D^{2\boldsymbol{e}} u\|_p \; \textstyle\sum_{l>n} 2^{-l} \sum_{|\boldsymbol{k}|=l} 2^{-\boldsymbol{k} \circ (\boldsymbol{e} - \boldsymbol{m})} \\
&\le C \, \|D^{2\boldsymbol{e}} u\|_p \, 2^{-n} \, (C_1 \, 2^{-n} n^{d-|\boldsymbol{m}|-1} + C_2 n^{|\boldsymbol{m}|-1}) \, ,
\end{aligned} \tag{5.16}$$

with $C_1 = 0$ if $|\boldsymbol{m}| = d$, and $C_2 = 0$ if $|\boldsymbol{m}| = 0$. Because

$$\|v\|_{\mathcal{W}_p^1} = \left(\|v\|_p^p + \sum_{|\boldsymbol{m}|=1, \boldsymbol{0} \le \boldsymbol{m} \le \boldsymbol{e}} \|D^{\boldsymbol{m}} v\|_p^p \right)^{1/p},$$

we consider the case $|\boldsymbol{m}| = 1$ and we find

$$\|u - \widehat{R}_n u\|_{W_p^1} \le C \, \|D^{2\boldsymbol{e}} u\|_p \, \|\boldsymbol{h}\| .$$

Together with the result of Theorem 5.2 this proves the theorem. □

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Summary of notations

We use the following notation for partial derivatives, $\boldsymbol{n}\in\mathbb{N}_0^d$,

$$D^{\boldsymbol{n}} = D^{n_1,\cdots,n_d} = \left(\frac{\partial}{\partial x_1}\right)^{n_1}\cdots\left(\frac{\partial}{\partial x_d}\right)^{n_d}.$$

For the Banach spaces of continuously differentiable functions we use the notations

$$C^{\boldsymbol{n}}(\Omega) = \{u \mid \max_{\mathbf{o}\le\boldsymbol{m}\le\boldsymbol{n}} \max_{x\in\Omega} |D^{\boldsymbol{m}}u(x)| < \infty\},$$

$$\|u\|_{C^{\boldsymbol{n}}} = \max_{\mathbf{o}\le\boldsymbol{m}\le\boldsymbol{n}} \max_{x\in\Omega} |D^{\boldsymbol{m}}u(x)|, \tag{5.1}$$

and, with $l \in \mathbb{N}_0$,

$$\mathcal{C}^l(\Omega) = \{u \in C(\Omega) \mid u\in C^{\boldsymbol{m}}(\Omega);\ |\boldsymbol{m}| \le l\}. \tag{5.2}$$

$$\|u\|_{\mathcal{C}^l} = \max_{|\boldsymbol{m}|\le l} \|u\|_{C^{\boldsymbol{m}}} = \max_{|\boldsymbol{m}|\le l} \max_{x\in\Omega} |D^{\boldsymbol{m}}u(x)|. \tag{5.3}$$

For the Banach spaces of integrable functions, $1 \le p \le \infty$, we, similarly, use the notation

$$W_p^{\boldsymbol{n}}(\Omega) = \left\{u \;\middle|\; \sum_{\mathbf{o}\le\boldsymbol{m}\le\boldsymbol{n}} \int_{x\in\Omega} |D^{\boldsymbol{m}}u(x)|^p < \infty\right\},$$

$$|u|_{W_p^{\boldsymbol{n}}} = \sqrt[p]{\int_{x\in\Omega} |D^{\boldsymbol{n}}u(x)|^p} \quad\text{and}\quad \|u\|_{W_p^{\boldsymbol{n}}} = \sqrt[p]{\sum_{\mathbf{o}\le\boldsymbol{m}\le\boldsymbol{n}} |u|^p_{W_p^{\boldsymbol{n}}}}.$$

For $0 \le k \le d$ we write $\mathcal{W}_p^{\boldsymbol{n},k}(\Omega) = \bigcap_{|\boldsymbol{m}|=k} W_p^{\boldsymbol{n}+\boldsymbol{m}}(\Omega)$. and

$$\|u\|_{\mathcal{W}_p^{\boldsymbol{n},k}} = \left(\sum_{\mathbf{o}\le\boldsymbol{m}\le\boldsymbol{e},|\boldsymbol{m}|=k} \|D^{\boldsymbol{n}+\boldsymbol{m}}u\|_p^p\right)^{1/p}.$$

With a $W_{p,0}^k(\Omega)$ and $\mathcal{W}_{p,0}^{\boldsymbol{n},k}(\Omega)$ we denote the corresponding subspaces with homogeneous boundary conditions. For $p = \infty$ we use the standard modifications, and for W_2 we also write H.

Table 1. Elementary notations

Symbol	Meaning
$\mathbb{N}$	natural numbers
$\mathbb{N}_0$	$\mathbb{N}\cup\{0\}$
$\mathbb{Z}$	integer numbers
$\mathbb{R}$	real numbers
$\mathbb{R}_+$	positive real numbers
$\boldsymbol{n}$	$(n_1,\cdots,n_d)$, multi-integer
$\lvert\boldsymbol{n}\rvert$	$n_1+n_2+\cdots+n_d$, $n\in\mathbb{N}_0^d$
$\lVert\boldsymbol{n}\rVert$	$n_1\cdot n_2\cdots n_d$, $n\in\mathbb{N}_0^d$
$\mathbf{o}\in\mathbb{N}_0^d$	$(0,\cdots,0)$
$\boldsymbol{e}_i\in\mathbb{N}_0^d$	$(0,\cdots,0,1,0,\cdots,0)$, the i-th unit vector
$\boldsymbol{e}\in\mathbb{N}_0^d$	$(1,1,\cdots,1)$
$\boldsymbol{E}$	$\{\boldsymbol{e}_1,\boldsymbol{e}_2,\cdots,\boldsymbol{e}_d\}$

Table 2. Grids

Symbol	Meaning
$\Omega \subset \mathbb{R}^d$	connected open subset
$\boldsymbol{h_n} \in \mathbb{R}^d_+$	$(h_{n_1}, h_{n_2}, \cdots, h_{n_d}) = (2^{-n_1}, 2^{-n_2}, \cdots, 2^{-n_d})$
$\lVert h_{\boldsymbol{n}} \rVert$	$\sqrt{\sum_{j=1}^d h_{n_j}^2}$, $\boldsymbol{h_n} \in \mathbb{R}^d_+$
$\lVert\lvert h_{\boldsymbol{n}} \rvert\rVert$	$\prod_{j=1}^d h_{n_j} = 2^{-\lvert \boldsymbol{n} \rvert}$, $\boldsymbol{h_n} \in \mathbb{R}^d_+$
$\Omega_{\boldsymbol{n},\boldsymbol{i}}$	$\{\boldsymbol{x} \mid \boldsymbol{i h_n} \le \boldsymbol{x} < (\boldsymbol{i}+\boldsymbol{e})\boldsymbol{h_n}\}$, a grid cell
$\Omega_{\boldsymbol{n}}$	$\{\Omega_{\boldsymbol{n},\boldsymbol{i}} \mid \Omega_{\boldsymbol{n},\boldsymbol{i}} \subset \Omega, \boldsymbol{i} \in \mathbb{Z}^d\}$, a grid of cells
$N_{\boldsymbol{n}} \subset \mathbb{R}^d$	$\{\boldsymbol{x} \mid \boldsymbol{x} = \boldsymbol{i h_n}, \boldsymbol{i} \in \mathbb{Z}^d\}$
$\Omega^+_{\boldsymbol{n}}$	$\Omega \cap N_{\boldsymbol{n}}$ a grid of (interior) vertex nodes
$\overline{\Omega}^+_{\boldsymbol{n}}$	$\overline{\Omega} \cap N_{\boldsymbol{n}}$ a grid of vertex nodes
$Z_{\boldsymbol{n}} \subset \mathbb{R}^d$	$\{\boldsymbol{z}_{\boldsymbol{n},\boldsymbol{i}} \mid \boldsymbol{z}_{\boldsymbol{n},\boldsymbol{i}} = (\boldsymbol{i}+\boldsymbol{e}/2)\boldsymbol{h_n}, \boldsymbol{i} \in \mathbb{Z}^d\}$
$\Omega^*_{\boldsymbol{n}}$	$\Omega \cap Z_{\boldsymbol{n}}$, a grid of cell centers

Table 3. Families of grids

Symbol	Meaning
G	$\{\Omega_{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{Z}^d\}$, infinite grid of grids
$G_{\boldsymbol{n}}$	$\{\Omega_{\boldsymbol{m}} \mid \mathbf{o} \le \boldsymbol{m} \le \boldsymbol{n},\ \boldsymbol{m} \in \mathbb{Z}^d\}$, a finite grid of grids
G^+	$\{\Omega^+_{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{Z}^d\}$
$G^+_{\boldsymbol{n}}$	$\{\Omega^+_{\boldsymbol{m}} \mid \mathbf{o} \le \boldsymbol{m} \le \boldsymbol{n},\ \boldsymbol{m} \in \mathbb{Z}^d\}$
G^*	$\{\Omega^*_{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{Z}^d\}$
$G^*_{\boldsymbol{n}}$	$\{\Omega^*_{\boldsymbol{m}} \mid \mathbf{o} \le \boldsymbol{m} \le \boldsymbol{n},\ \boldsymbol{m} \in \mathbb{Z}^d\}$

Table 4. Sparse grids

Symbol	Meaning
D_n	$\{\Omega_{\boldsymbol{m}} \mid \boldsymbol{m} \in \mathbb{Z}^d,\ \lvert\boldsymbol{m}\rvert \le n\}$, an infinite sparse grid family
$\overline{D}_n$	$\{\Omega_{\boldsymbol{m}} \mid \boldsymbol{m} \in \mathbb{Z}^d,\ \mathbf{o} \le \boldsymbol{m},\ \lvert\boldsymbol{m}\rvert \le n\}$, a (finite) sparse grid family
D^+_n	$\{\boldsymbol{x} \mid \boldsymbol{x} \in \Omega^+_{\boldsymbol{m}},\ \mathbf{o} \le \lvert\boldsymbol{m}\rvert \le n\}$, a sparse FEM grid
D^*_n	$\{\boldsymbol{x} \mid \boldsymbol{x} \in \Omega^*_{\boldsymbol{m}},\ \mathbf{o} \le \lvert\boldsymbol{m}\rvert \le n\}$, a sparse FVM grid

Table 5. Mappings

Symbol	Meaning
$X(\Omega)$	a Banach space of functions defined on Ω
$V_{\boldsymbol{n}}$	$\mathrm{Ran}(R_{\boldsymbol{n}})$
$R_{\boldsymbol{n}}$	$X(\Omega) \to V_{\boldsymbol{n}}$, a restriction
$P_{\boldsymbol{n}}$	$V_{\boldsymbol{n}} \to X(\Omega)$, a prolongation
$\Pi_{\boldsymbol{n}}$	$X \to \mathrm{Ran}(P_{\boldsymbol{n}}) \subset X$, a projection
$R_{\boldsymbol{mn}}$	$V_{\boldsymbol{n}} \to V_{\boldsymbol{m}}$, $\boldsymbol{m} \le \boldsymbol{n}$, a restriction
$P_{\boldsymbol{nm}}$	$V_{\boldsymbol{m}} \to V_{\boldsymbol{n}}$, $\boldsymbol{m} \le \boldsymbol{n}$, a prolongation

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