ERROR BOUNDS FOR EXPONENTIALLY FITTED GALERKIN METHODS APPLIED TO STIFF TWO-POINT BOUNDARY VALUE PROBLEMS

P.P.N. de Groen
University of Technology, Eindhoven, The Netherlands

P.W. Hemker
Mathematical Centre, Amsterdam, The Netherlands

ABSTRACT

A linear second order singularly perturbed two-point boundary-value problem is considered. Discretisation by means of Petrov-Galerkin methods of finite element type, where the trial spaces contain piecewise exponentials, is studied. Error bounds, both pointwise and in the energy norm, are derived. The relation with other special difference schemes is shown and the error bounds obtained are compared with numerical results.

1. INTRODUCTION

We study special Galerkin methods for computation of numerical approximations to the singularly perturbed boundary-value problem on the interval [a,b]

$$\begin{aligned} L_\varepsilon u &:= -\varepsilon u'' + pu' + qu = f, \quad (' = d/dx) \\ u(a) &= u(b) = 0 \end{aligned} \tag{1.1}$$

where ε is a small positive parameter and where p, q and f are sufficiently smooth functions which satisfy

$$\left.\begin{aligned} p(x) &\geq p_0 > 0 \\ q(x) - \tfrac{1}{2}p'(x) &\geq 1 \end{aligned}\right\} \quad \forall x \in [a,b]. \tag{1.2}$$

It is well known that $y_\varepsilon \in H_0^1(a,b)$ is a solution of problem (1.1) if and only if it is a solution of the Galerkin (or weak) form

$$\begin{cases} u \in H_0^1(a,b) \quad \text{and} \\ B_\varepsilon(u,v) := \varepsilon(u',v') + (pu'+qu,v) = (f,v) \\ \qquad\qquad \forall v \in H_0^1(a,b), \end{cases} \tag{1.3}$$

where $(\cdot,\cdot)$ denotes the usual innerproduct in $L^2(a,b)$. Moreover, both problems have a unique solution, which we shall denote by y_ε in the sequel.

By choosing in $H_0^1(a,b)$ subspaces S^h and V^h of equal finite dimension we obtain the Petrov-Galerkin discretisation of problem (1.1) : find $y_\varepsilon^h \in S^h$ such that

$$B_\varepsilon(y_\varepsilon^h,v) = (f,v) \quad \forall v \in V^h. \tag{1.4}$$

The space S^h is called the *solution space* and V^h the *test space*, whereas both spaces are called *trial spaces*.
For non-stiff two-point boundary value problems both the solution and the test space are usually chosen to be equal to the space P_k^h of piecewise polynomials of degree $\leq k$ on a quasi-uniform mesh Δ,

$$\Delta := \{x_i \mid i=0,1,\ldots,n\}, \quad a = x_0 < x_1 < x_2 <\ldots< x_n = b, \tag{1.5}$$

$$h_i := x_i - x_{i-1}, \quad h := \max_i h_i, \quad \min_i h_i/h \geq \mu > 0.$$

$$P_k^h := \{u \in H_0^1(a,b) \mid D^{k+1}u\,|_{(x_{i-1},x_i)} = 0\}, \tag{1.6}$$

where D stands for differentiation and $u_{|I}$ denotes the

restriction of the function u to the open interval I.
When such trial spaces are used for non-stiff problems, the Galerkin discretisation yields an approximation to the solution which is almost as good as the best approximation of the solution in the solution space. Moreover, the Galerkin approximation shows "superconvergence" at the mesh-points, since the test space contains good approximations of Green's function at the mesh-points (cf. Douglas and Dupont (1974)).

In our stiff problems, where ε is a small parameter (i.e. the ratio $hp(x)/\varepsilon$ is large), piecewise polynomial spaces (in general) do not contain satisfactory approximations to the solution and to Green's function. The reason is that the solution of (1.1) and Green's function have narrow boundary layers in which their slope is very large. In order to improve the approximation properties of the solution space we add to P_k^h in each subinterval a piecewise exponential that is a local approximation to the singular (i.e. the rapidly varying) solution of the equation $L_\varepsilon u = 0$. On the subinterval $[x_{i-1},x_i]$ the principal (singular) part of L_ε is $-\varepsilon D^2 + p(x_i)D$ whose singular solution is an increasing exponential. Therefore, with a non-negative "fitting function" $\alpha(x)$, we define a finite dimensional space E_k^h by

$$E_k^h := \{u \in H_0^1(a,b) \mid D^{k+1}(D-\alpha(x_i))u\,|_{(x_{i-1},x_i)} = 0,\quad i = 1,\dots,n\} \tag{1.7}$$

With $\alpha(x) \equiv p(x)/\varepsilon$, this space is fitted exponentially to the singular part of L_ε and it indeed contains a good approximation of the solution y_ε of (1.1).
Likewise we improve the approximation properties of the testspace by adding local approximations to the singular

solution of the adjoint equation $L_\varepsilon^* u = 0$. The principal singular part of L_ε^* on (x_{i-1},x_i) is $-\varepsilon D^2 - p(x_{i-1})D$, whose singular solution is an exponential decaying to the right. Therefore we define the finite dimensional space F_k^h by

$$F_k^h := \{u \in H_0^1(a,b) \mid D^{k+1}(D+\alpha(x_{i-1}))u \mid_{(x_{i-1},x_i)} = 0, \quad , i = 1,\ldots,n\} \qquad (1.8)$$

With $\alpha(x) \equiv p(x)/\varepsilon$, this space is fitted exponentially to the singular part of L_ε^* and it contains good approximations of $G_\varepsilon(x_i,\cdot)$, $i = 1,\ldots, n-1$, Green's function of (1.1) at the nodes.

The dimension of E_k^h and F_k^h is given by $\dim(E_k^h) = \dim(F_k^h) = nk + n - 1$. We see that $P_k^h \subset E_k^h \cap F_k^h$ for any fitting function α and we notice that both spaces E_k^h and F_k^h coincide if $\alpha(x_i) = 0$, $i = 0,1,2,\ldots,n$, in which case $E_k^h = F_k^h = P_{k+1}^h$. If $\alpha(x_i) \neq 0$ the space E_k^h contains the exponential $\exp(+\alpha(x_i)x)$ on (x_{i-1},x_i) and F_k^h contains the exponential $\exp(-\alpha(x_i)x)$ on (x_i,x_{i+1}).

In this paper we shall consider only exponentially fitted spaces with fitting function $\alpha(x) \equiv p(x)/\varepsilon$, which is the natural choice for a problem of type (1.4). With the aid of these spaces we obtain several different Petrov-Galerkin discretisations for problem (1.1). For each of these discretisations existence of a unique solution is guaranteed by an a priori estimate of the following type

$$\exists d > 0 \quad \forall u \in S^h \quad \exists v \in V^h : B_\varepsilon(u,v) \geq d \, \|u\|_\varepsilon \|v\|_\varepsilon , \qquad (1.9)$$

where $\|\cdot\|_\varepsilon$ denotes the energy-norm related to B_ε,

$$\|u\|_\varepsilon^2 := \varepsilon\|u'\|^2 + \|u\|^2. \qquad (1.10)$$

Error estimates for the solutions of the discretised problems can be derived both pointwise at the nodes and in the energy norm (see also De Groen (1978)). The orders of the error estimates are given in table 1.

TABLE 1

The order of the error estimates obtained for exponentially fitted Galerkin methods. The dimension of all trial spaces is nk + n - 1. *For comparison with the numerical experiments see table* 3.

	S_h	V_h	Order of the error		restrictions
			in the $\Vert\cdot\Vert_\varepsilon$ norm	at mesh points	in the proof
1	P_{k+1}	P_{k+1}	1	1	none
2	E_k	E_k	$\varepsilon+h^k$	$\varepsilon+h^k$	none
3	F_k	F_k	1	$\varepsilon+h^k$	none
4	$E_{k-1}+F_{k-1}$	$E_{k-1}+F_{k-1}$	$\varepsilon+h^{k-1}$	ε^2+h^{2k-2}	none
5	E_k	P_{k+1}	$\varepsilon+h^k$	$\varepsilon+h^k$	$h+\frac{\varepsilon}{h}<\gamma$
6	E_k	F_k	$\varepsilon+h^k$	ε^2+h^{2k}	$h+\frac{\varepsilon}{h}<\gamma$
7	P_{k+1}	F_k	1	$\varepsilon^2/h+h^{2k+1}$	$h+\frac{\varepsilon^2}{h}<\gamma$

The most remarkable of these results is 7, in which the solution space has no special virtues for approximation of the singular solution and in which nevertheless a high accuracy is obtained at the points of the mesh.

In section 2 of this paper we describe the construction of exponentially fitted finite element schemes and we show the relation to other difference schemes. In section 3 we give the proof of the error bounds for the cases $S^h = E_k^h$ and $S^h = P_{k+1}^h$, $V^h = F_k^h$. In section 4 we report results from numerical experiments and we compare them with the error bounds derived.

2. EXPONENTIALLY FITTED FINITE DIFFERENCE SCHEMES

In this section, first we describe sets of basis functions for exponentially fitted trial spaces which are suitable for computational purposes. Thereafter, using these basis functions, we give some examples of exponentially fitted finite element methods and, for some special cases, we compute the resulting difference schemes. Finally we show their relation to difference schemes as proposed by Il'in (1969) and Abrahamsson, Keller and Kreiss (1974).

(2a) *Basis functions in* E_k^h *and* F_k^h

Let $\{\phi_i \mid i = 1,\dots,m\}$ and $\{\psi_i \mid i = 1,\dots,m\}$ be bases in the solution space S^h and the testspace V^h respectively. Applying Petrov-Galerkin methods, we seek an approximation y_ε^h of the form

$$y_\varepsilon^h = \sum_{j=1}^{m} a_j \phi_j \ , \tag{2.1}$$

which satisfies the m equations

$$B_\varepsilon(y_\varepsilon^h \ , \psi_i) = (f,\psi_i), \ i = 1,\dots,m. \tag{2.2}$$

Hence, for actual construction of a Petrov-Galerkin discretisation, the selection of a proper set of basis functions is a major issue.

The following two practical considerations give an indication how to find suitable sets of functions $\{\phi_i\}$ and $\{\psi_i\}$.

1. If n-1 basis functions have the support (x_{i-1}, x_{i+1}) for $i = 1,2,\dots, n-1$ and the nk remaining basis functions have their support in a single subinterval only, then the resulting linear system is block-tridiagonal and can be reduced to a tridiagonal system by static condensation.
2. In order to obtain discretisations in which a subset $\{a_{m_i} \mid i = 1,\dots, n-1\}$ of the coefficients $\{a_j\}$ yields the values of the approximation y_ε^h at the nodes, one has to select the basis functions $\{\phi_j\}$ such that

$$\phi_j(x_i) = \delta_{j,m_i}, \quad 1 \le i \le n-1, \quad 1 \le j \le nk+n-1.$$

For k = 0 these considerations determine the basis functions in E_k^h and F_k^h uniquely because $\dim(E_0^h) = \dim(F_0^h) = n-1$ and there are n-1 values $y(x_i)$ to compute. The requirements

$$\begin{aligned}&\phi_j \in E_0^h\\ &\text{support}\,(\phi_j) \subset (x_{i-1},x_{i+1})\\ &\phi_i(x_i) = \delta_{ij}\end{aligned}$$

yield the set of basis functions $\{\phi_i\}_{i=1}^{n-1}$ in E_0^h;

$$\phi_i(x)=\begin{cases}1 - \Psi((x - x_{i-1})/h_i,\ \alpha_i h_i), & x \in (x_{i-1},x_i),\\ \Psi((x - x_i)/h_{i+1},\ \alpha_{i+1}h_{i+1}), & x \in (x_i,x_{i+1}),\\ 0, & x \notin (x_{i-1},x_{i+1}),\end{cases} \tag{2.3}$$

where we use the notations

$$\Psi(\xi,\alpha) := \frac{e^{\alpha\xi} - e^{\alpha}}{1 - e^{\alpha}} \quad \text{and } \alpha_i := \alpha(x_i) \ . \tag{2.4}$$

Analogously the basis functions in F_0^h are given by

$$\psi_i(x)=\begin{cases}1 - \Psi((x - x_{i-1})/h_i,\ -\alpha_{i-1}h_i), & x \in (x_{i-1},x_i),\\ \Psi((x - x_i)/h_{i+1},\ -\alpha_i h_{i+1}), & x \in (x_i,x_{i+1}),\\ 0, & x \notin (x_{i-1},x_{i+1}).\end{cases} \tag{2.5}$$

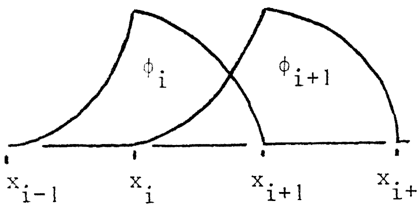

Fig. 1a.
Basis function ϕ_i *in* E_0^h.

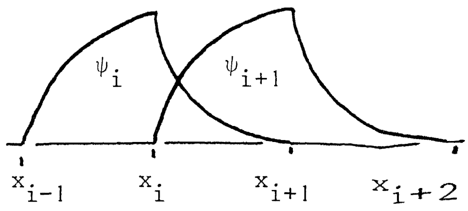

Fig. 1b.
Basis function ψ_i *in* F_0^h.

We notice that for $h/\varepsilon \to 0$ the exponentially fitted basis functions tend to the usual piecewise linear hat-functions and that for $h/\varepsilon \to \infty$, ϕ_i tends to the characteristic function of (x_i, x_{i+1}) and ψ_i tends to the characteristic function of (x_{i-1}, x_i).

For $k > 0$ there are several possibilities to form bases in E_k^h or F_k^h which satisfy the above mentioned two considerations.

(1.) We can extend the usual set of k-th degree C^o - piecewise polynomials which form a Lagrange type finite element basis in P_k^h to a basis for E_k^h or F_k^h. To complete the basis it should be supplemented by the exponential. For $k > 0$ we can find this exponential basis function with a support in a single interval by taking in (x_{i-1}, x_i) a linear combination of the exponential and a polynomial from P_k^h such that the resulting function vanishes at x_{i-1} and x_i

(1 A.) If this Lagrange type finite element basis in P_k^h on (x_{i-1}, x_i) is based on a subdivision

$$x_{i-1} = \xi_0 < \xi_1 < \ldots < \xi_k = x_i,$$

this polynomial can be taken such that the exponential basis function vasishes at $\xi_0, \xi_1, \ldots, \xi_k$.

(1 B.) This polymonial can also be taken linear such that the exponential basis function on (x_{i-1}, x_i) for E_k^h becomes

$$(\exp(\alpha_i x) - \exp(\alpha_i x_{i-1}))h_i +$$
$$- (\exp(\alpha_i x_i) - \exp(\alpha_i x_{i-1}))(x - x_{i-1}) \qquad (2.6)$$

and the exponential basis function for F_k^h on (x_{i-1},x_i) is

$$(\exp(-\alpha_{i-1} x) - \exp(-\alpha_{i-1} x_{i-1}))h_i +$$
$$- (\exp(-\alpha_{i-1} x_i) - \exp(-\alpha_{i-1} x_{i-1}))(x - x_{i-1}). \qquad (2.7)$$

Only in the case where $\alpha_i = 0$ or $\alpha_{i-1} = 0$ the functions (2.6) and (2.7) vanish on (x_{i-1},x_i), and have to be replaced by a (k+1)-th degree polynomial which vanishes at x_{i-1} and x_i.

(2.) Given a subdivision $x_{i-1} = \xi_0 < \xi_1 < \ldots < \xi_{k+1} = x_i$, another basis can be found in E_k^h by taking on (x_{i-1},x_i) a Lagrange-type polynomial base on $\xi_0, \xi_1, \ldots, \xi_k$ (polynomials that do not vanish at $\xi_{k+1} = x_i$), by adding the exponential function $1 - \Psi((x-x_{i-1})/h_i, \alpha_i h_i)$ and by correcting the k+1 polynomials by this exponential such that the resulting basis functions vanish at x_i (cf. Hemker (1977)).

Bases in F_k^h can be formed analogously.

(2b) *Exponentially fitted finite element / finite difference schemes*

With the above basis functions in the equations (2.1) and (2.2), the discretisation of the problem (1.1) leads to a block-tridiagonal linear system which, by static condensation, can be reduced to a tridiagonal system. The result is that a three-term difference scheme is obtained. For the general case the explicit description of such schemes is rather laborious. A full description of some of these schemes is given in Hemker (1977). In this paper we shall restrict ourselves to some simple examples which already show the main features of the more general and higher order methods.

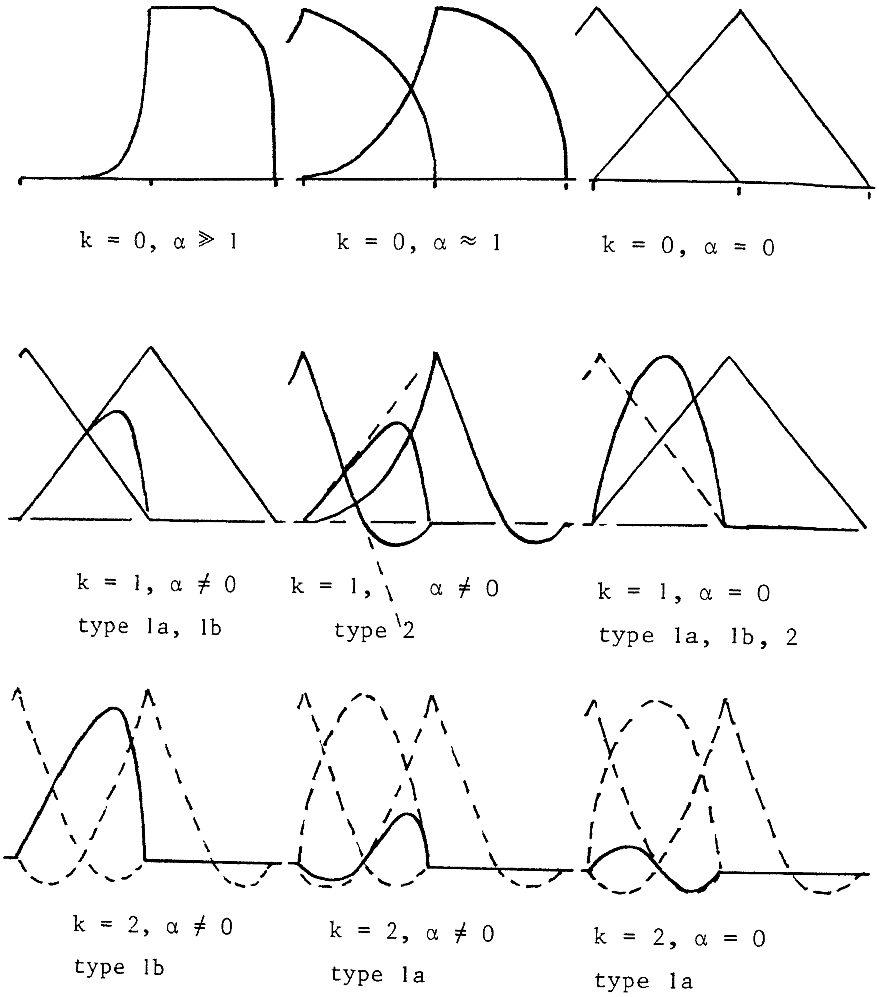

Fig. 2. *Several basis functions in* E_k^h, k = 0,1,2.

Example 1

If, for the discretisation of the model equation

$$-\varepsilon y'' + y' = 0, \tag{2.8}$$

with inhomogeneous boundary conditions and on a uniform mesh, we apply the Petrov-Galerkin method with the solution space $S^h = P_1^h$ and the testspace $V^h = F_0^h$ with $\alpha(x) - p(x)/\varepsilon = 1/\varepsilon$, then we obtain the difference scheme

$$\{ -\frac{\varepsilon}{h} - \frac{1}{2}(1+m)\}y_{i-1} + \{ \frac{2\varepsilon}{h} + m \}y_i + \\ + \{ -\frac{\varepsilon}{h} + \frac{1}{2}(1-m)\}y_{i+1} = 0, \tag{2.9}$$

where $m = \coth(\frac{h}{2\varepsilon}) - \frac{2\varepsilon}{h}$. This difference scheme is equivalent with Il'in's scheme, cf. Il'in (1969). In the limit for $h/\varepsilon \to 0$ it is equal to central differences and in the limit for $\varepsilon/h \to 0$ it is backward differences.

We remark that in this example the solution of the discretized problem is exact at the nodes, due to the fact that Green's function $G_\varepsilon(x_i,\cdot)$ of this problem is an element of the test space, cf. (3.44).

Example 2

If we apply the same Galerkin method as in the previous example to the constant coefficient equation

$$- \varepsilon y'' + py' + qy = f, \tag{2.10}$$

(i.e. we take $S^h = P_1^h$ and $V^h = F_0^h$ with $\alpha(x) \equiv p/\varepsilon$), then we obtain the following element stiffness matrix A and element loading vector b;

$$A := \frac{\varepsilon}{h}\begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} + \frac{p}{2}\begin{pmatrix} -1+m & 1-m \\ -1-m & 1+m \end{pmatrix} + \frac{qh}{4}\begin{pmatrix} 2-s-m & s-m \\ s+m & 2-s-m \end{pmatrix},$$

$$B := \frac{fh}{2}\begin{pmatrix} 1-m \\ 1+m \end{pmatrix}, \tag{2.11}$$

where $m := \coth\left(\frac{ph}{2\varepsilon}\right) - \left(\frac{2\varepsilon}{ph}\right)$ and

$$s := 1 - \frac{2\varepsilon}{ph}\, m = 1 - \frac{2\varepsilon}{ph}\left\{\coth\left(\frac{ph}{2\varepsilon}\right) - \left(\frac{2\varepsilon}{ph}\right)\right\} .$$

Note that

$$\lim_{\varepsilon/h \to 0} m = 1 \quad ; \quad \lim_{\varepsilon/h \to \infty} m = 0 \quad ;$$

$$\lim_{\varepsilon/h \to 0} s = 1 \quad ; \quad \lim_{\varepsilon/h \to \infty} s = 2/3.$$

Hence we obtain

$$\lim_{\varepsilon/h \to 0} A = p\begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} + \frac{qh}{2}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} ,$$

$$\lim_{\varepsilon/h \to 0} b = fh\begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Clearly, the reduced scheme reads

$$(-p + \frac{qh}{2})y_{i-1} + (p + \frac{qh}{2})y_i = fh , \qquad i = 1,2,\ldots, n-1. \qquad (2.12)$$

The same scheme is obtained by applying the trapezoidal rule to the reduced equation $pu' + qu = f$. For the constant coefficient equation the scheme (2.12) is equivalent with the box-scheme to which the method of Abrahamsson, Keller and Kreiss (1974) reduces for $\varepsilon \to 0$.

In the limit for $h/\varepsilon \to 0$ we obtain the scheme

$$(-\frac{\varepsilon}{h^2} - \frac{p}{2h} + \frac{q}{6})y_{i-1} + (\frac{2\varepsilon}{h^2} + \frac{2}{3}q)y_i + (-\frac{\varepsilon}{h^2} + \frac{p}{2h} + \frac{q}{6})y_{i+1} = f ,$$

which has also 2nd order accuracy.

For the non-constant coefficient equation the difference schemes contain integrals in which the coefficient functions p, q and f form part of the integrands. If the integrals are approximated by quadrature, the difference schemes obtained depend on the particular quadrature rule used.

Example 3

Now we discretize the model problem of example 1 on a uniform mesh by the Petrov-Galerkin method with $S^h = P_2^h$ and $V_h = F_1^h$ without prescribing the fitting function $\alpha(\mathbf{x})$ in advance. We obtain the element stiffness matrix

$$\begin{pmatrix} \frac{\varepsilon}{h} - \frac{1}{2} & \frac{1}{6} & -\frac{\varepsilon}{h} + \frac{1}{2} \\ -1 & R & 1 \\ -\frac{\varepsilon}{h} + \frac{1}{2} & -\frac{1}{6} & \frac{\varepsilon}{h} + \frac{1}{2} \end{pmatrix} \tag{2.13}$$

where

$$R := \frac{2\varepsilon}{h} + \frac{2}{\beta} + \left(\frac{6}{\beta} - 3\coth(\beta/2)\right)^{-1},$$

$$\beta := -\alpha(x_i)h.$$

If we apply exponential fitting (i.e. if we take $\alpha(x) \equiv 1/\varepsilon$), then $R = -1/3m$, where m is defined as in example 1. After static condensation this leads to the same difference scheme as in example 1, which yields the exact solution at the meshpoints.

If we consider the method in the limit for $h/\varepsilon \to 0$ (i.e. if we set $h\alpha(x) \equiv 0$), we obtain $R = \frac{2\varepsilon}{h}$ and after static condensation this leads to the 4-th order scheme:

$$\left[-\left(\frac{\varepsilon}{h} + \frac{h}{12\varepsilon}\right) - \frac{1}{2}\right]y_{i-1} + 2\left(\frac{\varepsilon}{h} + \frac{h}{12\varepsilon}\right)y_i +$$

$$+ \left[-\left(\frac{\varepsilon}{h} + \frac{h}{12\varepsilon}\right) + \frac{1}{2}\right]y_{i-1} = 0.$$

The latter scheme corresponds to the (2,2) – Padé approximation of $e^{h/\varepsilon}$ and hence shows no oscillations for $\varepsilon \to 0$, (cf. Van Veldhuizen, these proceedings).

3. RIGOROUS ERROR BOUNDS

In this section we shall derive rigourous error bounds for the Galerkin approximations $y_\varepsilon^h \in E_k^h$ and $\tilde{y}_\varepsilon^h \in P_{k+1}^h$ which satisfy the equations, cf. (1.4),

$$B_\varepsilon(y, v) = (f, v) \qquad \forall v \in F_k^h. \tag{3.1}$$

Error bounds for other combinations of solution and test spaces can be derived in the same way, cf. De Groen (1978). We have chosen these combinations, since they yield the best approximations. Moreover, the error bound for $\tilde{y}_\varepsilon^h$ is very remarkable; although the piecewise polynomial trial space P_{k+1}^h has no special virtues for approximation of the singular solution and although the error of $\tilde{y}_\varepsilon^h$ in the energy norm is of order unity, the error at the mesh-points is quite small.

We shall first sketch how a priory estimates and how error estimates for the best approximations in the trial spaces are obtained. Thereafter we shall give full proofs of the error estimates for y_ε^h and $\tilde{y}_\varepsilon^h$.

NOTE: C denotes a generic (positive) constant, which may differ on each occurrence; C may depend on the data a, b, f, p, q of the problem, the uniformity μ of the mesh, cf. (1.5) and on the degree k of the polynomials in the trial spaces. It certainly does not depend on ε and h.

(3a) *A priori estimates*

A priori estimates are used for comparison of the error of the Galerkin approximation with the error of the best approximation.

LEMMA 1:

$$\|u\|_\varepsilon^2 \le \|L_\varepsilon u\|^2 + |u(a)|^2 + |u(b)|^2, \quad \forall u \in H^2(a,b), \tag{3.2}$$

$$\|u\|_\varepsilon^2 \le B_\varepsilon(u,u), \qquad \forall u \in H_0^1(a,b), \tag{3.3}$$

$$B_\varepsilon(u,v) \le \left\{ \begin{array}{l} C\ \|u\|_\varepsilon \|v\|_1 \\ C\ \|u\|_1 \|v\|_\varepsilon \end{array} \right\} \le C\varepsilon^{-\frac{1}{2}} \|u\|_1 \|v\|_1, \qquad \forall u,v \in H_0^1(a,b) \tag{3.4}$$

PROOF: cf. De Groen (1978), lemmas 1,2. The inequalities are derived easily by integration by parts. □

In order to derive lower bounds for B_ε on $E_k^h \times F_k^h$ and on $P_{k+1}^h \times F_k^h$ we define the exponentials $w_i^\pm$ by

$$\begin{aligned} w_i^+(x) &:= \Psi((x-x_i)/h_i,\ \alpha_i h_i), \\ w_{i-1}^-(x) &:= \Psi((x-x_{i-1})/h_i,\ -\alpha_{i-1} h_i), \end{aligned} \tag{3.5}$$

where $\alpha_i := p(x_i)/\varepsilon$, cf. (2.4). They satisfy

$$D(\varepsilon D \mp p(x_i)) w_i^\pm = 0, \quad w_i^\pm(x_i) = 1, \quad w_i^\pm(x_{i\mp 1}) = 0. \tag{3.6}$$

The restriction to (x_{i-1},x_i) of an element $v \in F_k^h$ can be written as the sum of a polynomial π_i of degree $\le k$ plus a multiple of w_{i-1}^-,

$$v(x) = \pi_i(x) + \lambda_i w_{i-1}^-(x), \qquad \text{if } x_{i-1} \le x \le x_i. \tag{3.7}$$

For $v \in F_k^h$, decomposed in this way, and $x \in [x_{i-1},x_i]$ we define the maps $M^h : F_k^h \to E_k^h$ and $N_k^h : F_k^h \to P_{k+1}^h$ by

$$M^h v(x) = \pi_i(x) + \lambda_i(-1)^k\{P_k(\xi_i(x)) - w_i^+(x)\} \tag{3.8a}$$

$$N^h v(x) = \pi_i(x) + \tfrac{1}{2}\lambda_i(-1)^k\{P_k(\xi_i(x)) - P_{k+1}(\xi_i(x))\} \tag{3.8b}$$

where $\xi_i(x) := (2x-x_i-x_{i-1})/(x_i-x_{i-1})$ and where P_k stands for the k-th Legendre polynomial. By counting dimensions it is easily seen that the maps M^h and N^h are one-to-one from F_k^h onto E_k^h and P_{k+1}^k respectively. With the aid of these maps we find a priori estimates of type (1.9):

LEMMA 2: *A constant* $\gamma > 0$ *exists, such that*

$$\left.\begin{aligned} B_\varepsilon(M^h v,v) &\geq \tfrac{1}{2}\|v\|_\varepsilon \|M^h v\|_\varepsilon \qquad (3.9a)\\ B_\varepsilon(N^h v,v) &\geq \tfrac{1}{2}\|v\|_\varepsilon \|N^h v\|_\varepsilon \qquad (3.9b)\end{aligned}\right\} \forall v \in F^h_k,$$

provided $h + \varepsilon/h \leq \gamma$.

PROOF: Using the coercivity relation (3.3) we find

$$B_\varepsilon(M^h v,v) = B_\varepsilon(v,v) + B_\varepsilon(M^h v-v,v) \geq \|v\|^2_\varepsilon - |B_\varepsilon(M^h v-v,v)|.$$

Since $M^h v-v$ is zero at the mesh-points by definition, we may integrate the second term by parts,

$$B_\varepsilon(M^h v-v,v) = (M^h v-v,L^*_\varepsilon v).$$

Using the orthogonality properties of $P_k(\xi_i)$ on each subinterval separately we can show

$$|(M^h v-v,\ L^*_\varepsilon v)| \leq C(h+\varepsilon/h)^{\frac{1}{2}}\|v\|^2_\varepsilon.$$

Moreover, since we have the estimate

$$\|M^h v\|^2_\varepsilon \leq (1+Ch+C\varepsilon/h)\|v\|^2_\varepsilon\ , \qquad (3.10)$$

we can find a constant $\gamma > 0$, such that (3.9a) is true for all ε and h satisfying $h+\varepsilon/h < \gamma$. The proof of (3.9b) is analogous. For details we refer to De Groen (1978), lemmas 4 & 5. □

(3b) *Best approximations*

Best approximation of the solution y_ε of problem (1.1) in E^h_k and of Green's function G_ε in F^k_h are derived from asymptotic approximations, which are constructed by the method of "matched asymptotic expansions", cf. Eckhaus (1973) or O'Malley (1974).

The approximation of y_ε consists of a regular part (outer or regular expansion) and a singular part (boundary layer expansion). The lowest order terms $r_0 + \varepsilon r_1$ of the regular expansion are defined by the equations

$$pr_0'+qr_0 = f, \quad pr_1'+qr_1 = r_0'', \quad r_0(a) = r_1(a) = 0. \tag{3.11}$$

The lowest order terms $\tilde{s}_0 + \varepsilon\tilde{s}_1$ of the singular part are defined by

$$\tilde{s}_i(x) := s_i((x-b)/\varepsilon), \ (i=0,1) \text{ and } \zeta := (x-b)/\varepsilon,$$

$$-\ddot{s}_0+p(b)\dot{s}_0 = 0, \ -\ddot{s}_1+p(b)\dot{s}_1 = -\zeta p'(b)\dot{s}_0-q(b)s_0, \tag{3.12}$$

$$s_i(0) = -r_i(b), \ \lim_{\zeta \to -\infty} s_i(\zeta) = 0 \quad (i = 0,1).$$

We note that r' means differentiation with respect to the independent variable x and $\dot{s}$ means differentiation with respect to the boundary layer variable $\zeta := (x-b)/\varepsilon$. The equations (3.11-12) imply

$$\| f - L_\varepsilon (r_0+\varepsilon r_1)\| \le C\varepsilon^2, \ \| L_\varepsilon(\tilde{s}_0+\varepsilon\tilde{s}_1)\| \le C\varepsilon^{3/2} \tag{3.13}$$

and in conjunction with the a priori estimate (3.2) this yields

$$\| y_\varepsilon-r_0 -\varepsilon r_1 -\tilde{s}_0-\varepsilon\tilde{s}_1\|_\varepsilon \le C\varepsilon^{3/2}; \tag{3.14a}$$

from Sobolev's inequality $|u(x)| \le C\|u\|_1 \le C\varepsilon^{-\frac{1}{2}}\|u\|_\varepsilon$ we infer

$$\max_{a\le x\le b} |y_\varepsilon-r_0-\varepsilon r_1-\tilde{s}_0-\varepsilon\tilde{s}_1| \le C\varepsilon. \tag{3.14b}$$

Approximations of higher order may be derived analogously.

Likewise we construct an asymptotic approximation to Green's function $G_\varepsilon(x,\xi)$ for fixed $x \in (a,b)$. As a function of ξ it satisfies

$$\begin{aligned} L_\varepsilon^* G_\varepsilon(x,\cdot) &= \delta_x \ (= \text{Dirac's delta function}), \\ G_\varepsilon(x,a) &= G_\varepsilon(x,b) = 0, \end{aligned} \tag{3.15}$$

and it has boundary layers at the right-hand sides of the points $\xi = x$ and $\xi = a$. From a regular and a singular (approximate) solution of the equation $L_\varepsilon^* u = 0$ we construct a function whose derivative has the same jump at $\xi = x$ as

$G_\varepsilon(x,\cdot)$ has.

The regular approximate solution $\rho(x,\xi) := \rho_0(x,\xi) + \varepsilon\rho_1(x,\xi)$ is defined by

$$(p\rho_0)' - q\rho_0 = 0, \qquad \rho_0(x,x) = 1, \tag{3.16}$$
$$(p\rho_1)' - q\rho_1 = -\rho_0'', \quad \rho_1(x,x) = 0,$$

where the accent denotes differentiation with respect to ξ. Consequently ρ satisfies the estimate

$$\| L_\varepsilon^* \rho(x,\cdot) \|_{L^2(a,x)} \le C\varepsilon^2. \tag{3.17}$$

The singular approximate solution $\tilde{\sigma}(x,\xi)$, (the boundary layer term at $\xi = x+0$), is defined by

$$\begin{aligned} &\tilde{\sigma}(x,x+\varepsilon\zeta) := \sigma_0(x,\zeta) + \varepsilon\sigma_1(x,\zeta), \ \zeta := (\xi-x)/\varepsilon, \\ &\ddot{\sigma}_0 + p(x)\dot{\sigma}_0 = 0, \\ &\ddot{\sigma}_1 + p(x)\dot{\sigma}_1 = (q(x) - p'(x))\sigma_0 - \zeta p'(x)\dot{\sigma}_0, \\ &\sigma_0(x,0) = 1, \ \sigma_1(x,0) = 0, \ \lim_{\zeta\to\infty} \sigma_i(x,\zeta) = 0 \ (i = 1,2), \end{aligned} \tag{3.18}$$

where the dot denotes differentiation with respect to the boundary layer variable ζ (at $x + 0$). As a consequence of (3.18) we find the estimate

$$\| L_\varepsilon^* \tilde{\sigma}(x,\cdot) \|_{L^2(x,b)} \le C\varepsilon^{3/2}. \tag{3.19}$$

From these approximate solutions ρ and $\tilde{\sigma}$ we assemble an approximation of G_ε; its regular and singular parts G_ε^r and G_ε^s are defined by

$$\begin{aligned} G_\varepsilon^r(x,\xi) &:= -\beta\tilde{\sigma}(x,b)\rho(b,\xi) + \begin{cases} 0, & \text{if } x < \xi \le b, \\ \beta\rho(x,\xi), & \text{if } a \le \xi < x, \end{cases} \\ G_\varepsilon^s(x,\xi) &:= \beta\tilde{\sigma}(a,\xi)\{\rho(b,a)\tilde{\sigma}(x,b) - \rho(x,a)\} + \\ &\qquad + \begin{cases} \beta\tilde{\sigma}(x,\xi), & \text{if } x < \xi \le b, \\ 0, & \text{if } a \le \xi < x. \end{cases} \end{aligned} \tag{3.20}$$

It is easily seen that the sum $G_\varepsilon^r + G_\varepsilon^s$ is continuous at $\xi = x$. The multiplier β is chosen such that the jump of the ξ - derivative of $G_\varepsilon^r + G_\varepsilon^s$ at $\xi = x$ is equal to $1/\varepsilon$. Simple computation shows

$$\beta(x,\varepsilon) = 1/p(x) + \mathcal{O}(\varepsilon). \tag{3.21}$$

From (3.2) and (3.17-19-21) we find the estimate

$$\| G_\varepsilon(x,\cdot) - G_\varepsilon^r(x,\cdot) - G_\varepsilon^s(x,\cdot)\|_\varepsilon \leq C\varepsilon^{3/2} \tag{3.22}$$

uniformly with respect to $x \in [a,b]$.

From these asymptotic approximations we construct approximations which are in the exponentially fitted trial spaces E_k^h and F_k^h . Comparison of these approximations with the Galerkin approximation finally yields the desired error estimate for the latter. In order to obtain the highest possible order with respect to ε we have to deal with the regular and singular parts separately.

The regular approximation $r_0 + \varepsilon r_1$ of y_ε is non-zero at $x = b$, so we look for an approximation of it in the inhomogeneous linear manifold $\phi_\varepsilon + E_k^h$, where ϕ_ε is the linear polynomial

$$\phi_\varepsilon(x) := (r_0(b) + \varepsilon r_1(b))(x-a)/(b-a).$$

Well-known interpolation theorems imply that an approximation r_ε^h exists, such that

$$r_\varepsilon^h \in \phi_\varepsilon + P_k^h \subset \phi_\varepsilon + E_k^h.$$

$$\| r_\varepsilon^h - r_0 - \varepsilon r_1\|_0 \leq Ch^{k+1}, \quad \| r_\varepsilon^h - r_0 - \varepsilon r_1\|_1 \leq Ch^k. \tag{3.23}$$

Likewise the approximation S_ε^h of the singular part $\tilde{s}_0 + \varepsilon\tilde{s}_1$ has to be in $-\phi_\varepsilon + E_k^h$; for $x \in [x_{i-1},x_i]$ we define it by

$$s_\varepsilon^h(x) := -\phi_\varepsilon(b)\{\exp(\alpha_i(x-x_{i-1}))\prod_{j=i}^{n}\exp(-\alpha_j h_j) + \\ - \frac{b-x}{b-a}\prod_{j=1}^{n}\exp(-\alpha_j h_j)\}, \tag{3.24}$$

where $\alpha_j := p(x_j)/\varepsilon$, cf. (1.9). It is easily seen that this approximation satisfies

$$\| s_\varepsilon^h - \tilde{s}_0 - \varepsilon\tilde{s}_1 \|_\varepsilon \le C\varepsilon, \tag{3.25}$$

$$\| (-\varepsilon D+p)(s_\varepsilon^h - \tilde{s}_0 - \varepsilon\tilde{s}_1)\|_\varepsilon \le C\varepsilon^{3/2}. \tag{3.26}$$

By linearity $r_\varepsilon^h + s_\varepsilon^h \in E_k^h$ is a good approximation of y_ε.

Analogously we construct approximations to the regular and singular parts of Green's function. Since the derivative of $G_\varepsilon(x,\cdot)$ has a jump at $\xi = x$, we can find a satisfactory approximation in the space (of piecewise smooth functions) F_k^h only if this jump happens to coincide with a mesh-point. The regular approximation $G_\varepsilon^r(x,\cdot)$ has a jump at $\xi = x$ and is non-zero at $\xi = a$ and $\xi = b$, hence we construct approximations to it in $\psi_i + F_k^h$, where ψ_i is the piecewise linear polynomial $(i = 1,\dots,n-1)$

$$\psi_i(\xi) := \begin{cases} (G_\varepsilon^r(x_i,a)(\xi-x_i) - G_\varepsilon^r(x_i,x_i-0)(\xi-a))/(a-x_i) & \text{if } a \le \xi < x_i, \\ (G_\varepsilon^r(x_i,b)(\xi-x_i) - G_\varepsilon^r(x_i,x_i+0)(\xi-b))/(b-x_i) & \text{if } x_i < \xi \le b. \end{cases} \tag{3.27}$$

Analogously to above we find approximations

$$\rho_{\varepsilon,i}^h \in \psi_i + F_h^k \quad \text{and} \quad \sigma_{\varepsilon,i}^h \in -\psi_i + F_k^h,$$

which for $i = 1,\dots,n-1$ satisfy the estimates

$$\| G_\varepsilon(x_i,\cdot) - \rho^h_{\varepsilon,i} - \sigma^h_{\varepsilon,i} \|_\varepsilon \le C(\varepsilon + h^{k+1}), \tag{3.28}$$

$$\| (Dp-q)(\rho^h_{\varepsilon,i} - G^r_\varepsilon(x_i,\cdot)) \| \le Ch^k, \tag{3.29}$$

$$\| (\varepsilon D+p)(\sigma^h_{\varepsilon,i} - G^s_\varepsilon(x_i,\cdot)) \| \le C\varepsilon^{3/2}. \tag{3.30}$$

(3c) *Error estimates for the Galerkin approximations*

From the approximations constructed above we derive the following theorem:

THEOREM 1: *Let* $y^h_\varepsilon \in E^h_k$ *be the solution of the Galerkin equations*

$$B_\varepsilon(y,v) = (f,v) \qquad \forall v \in F^h_k. \tag{3.31}$$

If $h + \varepsilon/h \le \gamma$, *then* y^h_ε *satisfies the global estimate*

$$\| y_\varepsilon - y^h_\varepsilon \|_\varepsilon \le C(\varepsilon + h^k) \tag{3.32}$$

and it is superconvergent at the nodes,

$$|y^h_\varepsilon(x_i) - y_\varepsilon(x_i)| \le C(\varepsilon^2 + h^{2k}), \; i = 1,\dots,n-1. \tag{3.33}$$

PROOF: We shall derive error estimates for the regular and the singular part of y^h_ε separately. Let $u_\varepsilon \in \phi_\varepsilon + H^1_0(a,b)$ be the solution of

$$L_\varepsilon u_\varepsilon = f, \; u_\varepsilon(a) = 0, \; u_\varepsilon(b) = r_0(b) + \varepsilon r_1(b) \tag{3.34}$$

and let $u^h_\varepsilon \in \phi_\varepsilon + E^h_k$ satisfy the Galerkin equations (3.31) for this problem. Let $z_\varepsilon \in -\phi_\varepsilon + H^1_0(a,b)$ be the solution of

$$L_\varepsilon z_\varepsilon = 0, \; z_\varepsilon(a) = 0, \; z_\varepsilon(b) = -r_0(b) - \varepsilon r_1(b)$$

and let $z^h_\varepsilon \in -\phi_\varepsilon + E^h_k$ satisfy the Galerkin equations for this problem,

$$B_\varepsilon(z,v) = 0 \qquad \forall v \in F^h_k.$$

Linearity implies $u_\varepsilon + z_\varepsilon = y_\varepsilon$ and $u^h_\varepsilon + z^h_\varepsilon = y^h_\varepsilon$. Formulae (3.11-12-13) imply

$$\| u_\varepsilon - r_0 - \varepsilon r_1 \| \le C\varepsilon^2, \ \| z_\varepsilon - \tilde{s}_\varepsilon - \varepsilon \tilde{s}_1 \|_\varepsilon \le C\varepsilon^{3/2}; \qquad (3.35)$$

clearly u_ε and z_ε represent (in first order) the regular and singular parts of y_ε.

An error bound for u_ε^h is obtained by comparing it with r_ε^h, cf. (3.23). Since $u_\varepsilon^h - u_\varepsilon$ satisfies

$$B_\varepsilon(u_\varepsilon^h - u_\varepsilon, v) = 0 \qquad \forall v \in F_k^h,$$

we find

$$B_\varepsilon(u_\varepsilon^h - r_\varepsilon^h, v) = B_\varepsilon(u_\varepsilon - r_0 - \varepsilon r_1, v) + B_\varepsilon(r_0 + \varepsilon r_1 - r_\varepsilon^h, v). \qquad (3.36)$$

Using (3.4-35) we estimate the first term,

$$B_\varepsilon(u_\varepsilon - r_0 - \varepsilon r_1, v) \le C\varepsilon^{3/2} \| v \|_\varepsilon$$

and using (3.23) we find for the second term

$$B_\varepsilon(r_0 + \varepsilon r_1 - r_\varepsilon^h, v) \le Ch^k \| v \|_\varepsilon . \qquad (3.37)$$

Hence, lemma 2 and the choice $M^h v := u_\varepsilon^h - r_\varepsilon^h$ yield the estimate

$$\| u_\varepsilon^h - r_\varepsilon^h \|_\varepsilon \le C\left(\varepsilon^{3/2} + h^k\right), \quad (\text{if } h + \varepsilon/h \le \gamma). \qquad (3.38)$$

Likewise an error bound for z_ε^h is obtained by comparing z_ε^h and s_ε^h, cf. (3.24). Since $z_\varepsilon^h - z_\varepsilon$ satisfies

$$B_\varepsilon(z_\varepsilon^h - z_\varepsilon, v) = 0 \qquad \forall v \in F_k^h,$$

we find

$$B_\varepsilon(z_\varepsilon^h - s_\varepsilon^h, v) = B_\varepsilon(z_\varepsilon - \tilde{s}_0 - \varepsilon \tilde{s}_1, v) + B_\varepsilon(\tilde{s}_0 + \varepsilon \tilde{s}_1 - s_\varepsilon^h, v). \qquad (3.39)$$

From (3.4-35) we find

$$B_\varepsilon(z_\varepsilon - \tilde{s}_0 - \varepsilon \tilde{s}_1, v) \le C\varepsilon \| v \|_\varepsilon . \qquad (3.40)$$

For the second term in (3.39) we use the estimate

$$B_\varepsilon(u,v) = (\varepsilon u' - pu, v') + ((q-p')u, v) \le$$

$$\le \| v \|_\varepsilon \{ \varepsilon^{-\frac{1}{2}} \| u' - pu \| + C \| u \| \} \quad \forall u,v \in H_0^1(a,b). \qquad (3.41)$$

In conjunction with (3.26) this implies

$$B_\varepsilon(\tilde{s}_0+\varepsilon\tilde{s}_1-s_\varepsilon^h,v) \le C\varepsilon\|v\|_\varepsilon \quad \forall v \in F_k^h.$$

Hence, lemma 2 and the choice $M^h v := z_\varepsilon^h - s_\varepsilon^h$ yield the estimate

$$\|z_\varepsilon^h - s_\varepsilon^h\|_\varepsilon \le C\varepsilon, \quad (\text{if } h+\varepsilon/h \le \gamma). \tag{3.42}$$

We remark that this estimate does not depend on the degree of the polynomials in E_k^h. Formulae (3.38-42) imply (3.32).

In order to prove the superconvergence we use the identity

$$y(x) = B_\varepsilon(y,G_\varepsilon(x,\cdot)) \quad \forall y \in H_0^1(a,b),\ a < x < b. \tag{3.43}$$

Clearly, the error $e_\varepsilon^h := y_\varepsilon^h - y_\varepsilon$ satisfies the equations

$$e_\varepsilon^h(x) = B_\varepsilon(e_\varepsilon^h,G_\varepsilon(x,\cdot)) = B_\varepsilon(e_\varepsilon^h,G_\varepsilon(x,\cdot)-v)\ \forall v \in F_k^h, \tag{3.44}$$

cf. Douglas & Dupont (1974). If x is a node, F_k^h contains a good approximation of Green's function $G_\varepsilon(x,\cdot)$, namely $\rho_\varepsilon^h + \sigma_\varepsilon^h$. Hence, for $i = 1,\dots,n-1$, formula (3.44) implies

$$e_\varepsilon^h(x_i) = B_\varepsilon(e_\varepsilon^h,G_\varepsilon-G_\varepsilon^r-G_\varepsilon^s) + B_\varepsilon(e_\varepsilon^h,G_\varepsilon^r-\rho_{\varepsilon,i}^h) + B_\varepsilon(e_\varepsilon^h,G_\varepsilon^s-\sigma_{\varepsilon,i}^h). \tag{3.45}$$

Formulae (3.4-22) yield an estimate for the first term:

$$|B_\varepsilon(e_\varepsilon^h,G_\varepsilon-G_\varepsilon^r-G_\varepsilon^s)| \le C\varepsilon\|e_\varepsilon^h\|_\varepsilon. \tag{3.46}$$

Since $G_\varepsilon^r - \rho_{\varepsilon,i}^h$ and $G_\varepsilon^s - \sigma_{\varepsilon,i}^h$ both are in $H_0^1(a,b)$ by definition, we can use for the former the estimate

$$|B_\varepsilon(y,v)| \le \varepsilon\|y'\|\|v'\| + \|(Dp-q)y\|\|v\| \tag{3.47}$$

and for the latter the estimate

$$|B_\varepsilon(y,v)| \le \|\varepsilon v' + pv\|\|y'\| + \|qy\|\|v\|. \tag{3.48}$$

Hence, by (3.29-30) we find

$$\begin{aligned} |B_\varepsilon(e_\varepsilon^h,G_\varepsilon^r(x_i,\cdot) - \rho_{\varepsilon,i}^h)| &\le Ch^k\|e_\varepsilon^h\|_\varepsilon, \\ |B_\varepsilon(e_\varepsilon^h,G_\varepsilon^s(x_i,\cdot) - \sigma_{\varepsilon,i}^h)| &\le C\varepsilon\|e_\varepsilon^h\|_\varepsilon. \end{aligned} \tag{3.49}$$

The formulae (3.32-45-46-49) now imply the superconvergence (3.33). □

In an analogous fashion we derive:

THEOREM 2: *Let* $\tilde{y}_\varepsilon^h \in P_{k+1}^h$ *be the solution of* (3.31). *If* $h+\varepsilon/h < \gamma$, $\tilde{y}_\varepsilon^h$ *satisfies for* $i=1,\ldots,n-1$ *at the mesh points the estimates*

$$|\tilde{y}_\varepsilon^h(x_i) - y_\varepsilon(x_i)| \le \begin{cases} C(\varepsilon+h^k), \\ C(h^{2k+1} + \frac{\varepsilon^2}{h}(1+\frac{\varepsilon^{\frac{1}{2}}}{h})); \end{cases} \tag{3.50}$$

the second estimate is valid only if $\varepsilon|\log \varepsilon| < p_0 h$.

PROOF: Although the error of $\tilde{y}_\varepsilon^h$ in energy norm is of order unity, the error at the mesh points is of order $O(\varepsilon+h^k)$, since the test space contains an approximation of Green's function of that order. If ε/h is small enough, the $O(1)$-error in energy norm results from the poor approximation of the singular part of y_ε only. It is committed almost completely in the subinterval (x_{n-1},b), where it is cancelled by the smallness of Green's function. Thus we can improve the estimate at the nodes.

Let u_ε and z_ε be the regular and singular parts of the solution as defined in (3.34) and let $\tilde{u}_\varepsilon^h \in \phi_\varepsilon + P_{k+1}^h$ and $\tilde{z}_\varepsilon^h \in -\phi_\varepsilon + P_{k+1}^h$ be their Galerkin approximations,

$$B_\varepsilon(\tilde{u}_\varepsilon^h,v) = (f,v),\ B_\varepsilon(\tilde{z}_\varepsilon^h,v) = 0, \quad \forall v \in F_k^h.$$

Let $\tilde{r}_\varepsilon^h \in \phi_\varepsilon + P_{k+1}^h$ interpolate $r_0 + \varepsilon r_1$, such that

$$\|\tilde{r}_\varepsilon^h - r_0 - \varepsilon r_1\|_1 \le Ch^{k+1},$$

analogously to (3.23). Analogously to (3.38) we find

$$\|\tilde{u}_\varepsilon^h - \tilde{r}_\varepsilon^h\|_\varepsilon \le C(\varepsilon^{3/2}+h^{k+1}), \text{ if } h+\varepsilon/h \le \gamma.$$

Inserting this estimate in (3.45-46-49) we find for the regular part the error estimate

$$|u_\varepsilon(x_i) - \tilde{u}^h_\varepsilon(x_i)| \le C(\varepsilon^{3/2}+h^{k+1})(\varepsilon+h^k), \tag{3.51}$$

for $i = 1,\dots,n-1$, provided $h+\varepsilon/h \le \gamma$.

The error estimate for the singular part $\tilde{z}^h_\varepsilon$ is more involved since the solution space $-\phi_\varepsilon + P^h_k$ does not contain an approximation of z_ε, whose error is better than $O(1)$ in energy norm. We start with a preliminary error estimate at the knots by the superconvergence trick. Thereafter we improve this estimate by considering the errors on the subintervals $I := (a,x_{n-1})$ and $J := (x_{n-1},b)$ separately. We shall denote the restrictions of $\|\cdot\|_\varepsilon$ and B_ε to I and J by $\|\cdot\|_{\varepsilon,I}$, $\|\cdot\|_{\varepsilon,J}$, $B_{\varepsilon,I}$ and $B_{\varepsilon,J}$ respectively.

Using lemma 2 with $v := (M^h)^{-1}(\tilde{z}^h_\varepsilon+\phi_\varepsilon)$ we find

$$\tfrac{1}{2}\|\tilde{z}^h_\varepsilon + \phi_\varepsilon\|_\varepsilon \|v\|_\varepsilon \le B_\varepsilon(\tilde{z}^h_\varepsilon+\phi_\varepsilon,v) = B_\varepsilon(\phi_\varepsilon,v) \le C\|v\|_\varepsilon \,;$$

hence $\|\tilde{z}^h_\varepsilon\|_\varepsilon = O(1)$. Analogously we find $\|z_\varepsilon\|_\varepsilon = O(1)$. In the same way as in (3,44-48-49) we find from these energy norm estimates the pointwise estimate

$$|\tilde{z}^h_\varepsilon(x_i) - z_\varepsilon(x_i)| \le C(\varepsilon+h^k)\|\tilde{z}^h_\varepsilon - z_\varepsilon\|_\varepsilon \le C(\varepsilon+h^k). \tag{3.52}$$

If $(b-x_{n-1})p_0 > \varepsilon|\log\varepsilon|$, ($p_0$ as in (1.2)), the boundary layer is contained in the subinterval (x_{n-1},b) entirely and we have

$$|z_\varepsilon(x_{n-1})| \le C\varepsilon, \text{ hence } \zeta_\varepsilon := \tilde{z}^h_e(x_{n-1}) \le C(\varepsilon+h^k). \tag{3.53}$$

In P^h_{k+1} we now define the function w_ε by

$$w_\varepsilon := \begin{cases} \theta\{P_{k+1}(\xi_n)+P_k(\xi_n)+\eta h(P_k(\xi_n)+P_{k-1}(\xi_n))\} + \\ \qquad - (-1)^k\zeta_\varepsilon P_{k+1}(\xi_n), \quad \text{if } x \in J \\ \\ \zeta_\varepsilon(x-a)/(x_{n-1}-a), \qquad \text{if } x \in I, \end{cases} \tag{3.54}$$

where $\xi_n := (2x-x_n-x_{n-1})/(x_n-x_{n-1})$, cf. (3.8), and where θ and η are defined by

$$\eta := \frac{2k+1}{2k-1}\,\frac{(k+1)p'(b)+q(b)}{p(b)}\ ,\quad \theta := \frac{\phi_\varepsilon(b)+(-1)^k\zeta_\varepsilon}{2+2h\eta}\ .$$

On I we find the estimates

$$\|w_\varepsilon\|_{1,I} \le C(\varepsilon+h^k) \quad \text{and}$$

$$B_{\varepsilon,I}(w_\varepsilon,v) \le C(\varepsilon+h^k)\|v\|_\varepsilon\ ,\ \forall v \in F_k^h.$$

Hence $\|w_\varepsilon-z_\varepsilon\|_{\varepsilon,I}$ and $\|w_\varepsilon-\tilde{z}_\varepsilon^h\|_{\varepsilon,I}$ both are of the order $\mathcal{O}(\varepsilon+h^k)$ and this implies, cf. (3.38-42-49),

$$\|\tilde{z}_\varepsilon^h-z_\varepsilon\|_{\varepsilon,I} \le C(\varepsilon+h^k) \quad \text{and}$$

$$B_{\varepsilon,I}(\tilde{z}_\varepsilon^h-z_\varepsilon,\ G_\varepsilon(x_i,\cdot)-\rho_{\varepsilon,i}^h-\sigma_{\varepsilon,i}^h) \le C(\varepsilon+h^k)^2, \qquad (3.55)$$

$$i = 1,\dots,n-1.$$

On J we find the estimates

$$\|w_\varepsilon\|_{\varepsilon,J} \le C(h+\varepsilon/h)^{\frac{1}{2}} \quad \text{and}$$

$$B_{\varepsilon,J}(w_\varepsilon,v) \le C(h+\varepsilon/h)^{\frac{1}{2}}\|v\|_\varepsilon\ ,\ \forall v \in F_k^h;$$

in the proof of the second estimate we use the same trick as in the proof of lemma 2. These estimates imply

$$B_\varepsilon(w_\varepsilon-\tilde{z}_\varepsilon^h,v) \le C(h+\varepsilon/h),\ \forall v \le F_k^h\ .$$

In conjunction with lemma 2 we infer

$$\|\tilde{z}_\varepsilon^h\|_{\varepsilon,J} \le C(h+\varepsilon/h)^{\frac{1}{2}}$$

and since $\tilde{z}_\varepsilon^h$ is a polynomial it satisfies the estimate

$$|\tilde{z}_\varepsilon^h(x)| + h\left|\frac{d}{dx}\tilde{z}_\varepsilon^h(x)\right| \le C(1+\varepsilon^{\frac{1}{2}}/h),$$

$$|\tilde{z}_\varepsilon^h(x)| \le C(\varepsilon+h^k + \left|\frac{d}{dx}\tilde{z}^h(x_{n-1})\right|\,|x-x_{n-1}|).$$

Straightforward computation now yields

$$B_{\varepsilon,J}\left(\tilde{z}_\varepsilon^h-z_\varepsilon,\ G_{\varepsilon,i}-\rho_{\varepsilon,i}^h-\sigma_{\varepsilon,i}^h\right) \le C\left(h^{2k}+\frac{\varepsilon^2}{h}(1+\frac{\varepsilon}{h}^{\frac{1}{2}})\right). \qquad (3.56)$$

In conjunction with (3.55) this yields the estimate

$$|\tilde{z}_\varepsilon^h(x_i) - z_\varepsilon(x_i)| \le C\left(h^{2k} + \frac{\varepsilon}{h}^2 (1+\frac{\varepsilon}{h}^{\frac{1}{2}})\right) \qquad (3.57)$$

for $i = 1,\ldots,n-1$. If the term h^{2k} is dominant in this error estimate, it can be improved by repeating the process from formula (3.53) on, using the better estimate (3.57) instead of (3.52). So we obtain the desired estimate (3.50). □

4. RESULTS OF NUMERICAL EXPERIMENTS

Several numerical experiments were performed with the exponentially fitted methods described in the previous sections. The accuracy of the computed solution is considered at the mesh-points and this accuracy is compared with the error bounds derived.

(4a) *The experiments*

For the trial spaces S^h and V^h the spaces P^h_{k+1}, E^h_k, F^h_k and $E^h_{k-1} + F^h_{k-1}$ were used with $k = 1,2,3$. With these spaces the seven combinations for the solution and test space were used as they are mentioned in table 1. The partition of the interval of integration was taken quasi-uniformly with $n = 4$ and $n = 8$. For different values of h and ε the accuracy obtained at the mesh-points was compared in order to determine the dependence of the error on these two parameters. For ε the following sequence was used:

$$\varepsilon = 1,\ 0.1,\ 10^{-2},\ 10^{-3},\ 10^{-4},\ 10^{-5},\ 10^{-10}.$$

Mesh selection

In order to eliminate effects possibly due to a uniform partition, the experiments were done with non-uniform partitions, where the mesh-points were selected by

$$x_i = ih + 0.15\ \rho h, \quad i = 1,2\ldots,n-2,$$

where $h = 1/n$ and ρ is a random variable distributed uniformly in $[-1,+1]$.

Quadrature

In the experiments the computation of the integrals was executed by means of an automatic quadrature routine which computed the integrals with an absolute or relative accuracy of 10^{-7} on each subinterval of the grid separately. We are convinced that automatic quadrature is not an efficient procedure. However, since our purpose is to compare the error bounds derived with the actual errors for the methods described, we do not want to consider effects introduced by numerical quadrature. Hence we approximate the exact value of all the integrals involved as good as possible. For efficient quadrature techniques for the exponentially fitted methods we refer to Hemker (1977).

The environment

The experiments were performed in single precision on a CDC-CYBER computer, using the CDC ALGOL 68 compiler (version 1.2.0). The accuracy of a real number is about 14 decimal digits.

The problems

The following three problems were used in the experiments.

PROBLEM 1:

$$-\varepsilon y'' + (2+\cos(\pi x))y' + y = (1+\varepsilon\pi^2)\cos(\pi x) - (2+\cos(\pi x))\pi\sin(\pi x),$$

$$y(0) = 1, \qquad y(1) = -1.$$

The solution is $y(x) = \cos(\pi x)$; the solution has no boundary layer.

PROBLEM 2:

$$-\varepsilon y'' + y' + (1+\varepsilon)y = 0,$$
$$y(0) = 1 + \exp(-(1+\varepsilon)/\varepsilon),$$
$$y(1) = 1 + \exp(-1).$$

The solution is $y(x) = \exp((1+\varepsilon)(x-1)/\varepsilon) + \exp(-x)$; the equation has constant coefficients and the solution has a boundary layer.

PROBLEM 3:

$$-\varepsilon y'' + \cos(\alpha-x)y' + y = \sin(\alpha-x)(1+\varepsilon+\sin(\alpha-x)) - 1 + \exp((x-1)/\varepsilon)(1-2\sin((\alpha-x)/2)^2/\varepsilon),$$
$$y(0) = \sin(\alpha) + \exp(-1/\varepsilon),$$
$$y(1) = \sin(\alpha-1) + 1.$$

The solution is $y(x) = \sin(\alpha-x) + \exp((x-1)/\varepsilon)$. The equation has non-constant coefficients and the solution has a boundary layer at $x = 1$. In order to prevent results which may be flattered because $p'(b) = 0$, we have experimented both with $\alpha = 1$ and with $\alpha = 5\pi/12$, which imply $p'(1) = 0$ and $p'(1) \neq 0$ respectively.

(4b) *The numerical results*

In order to give an impression of the actual accuracy of the methods we give some examples of the results obtained for problem 3 ($\alpha = \frac{5}{12}\pi$) in table 2.

A summary of a complete series of experimental results is given in table 3 and 4.

TABLE 2

Errors at the meshpoints: $\max_{i=0,\ldots,n} |y_\varepsilon(x_i) - y_\varepsilon^h(x_i)|$.

h \ ε	1	0.1	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-10}
1/4	3.6 (-7)	1.4 (-3)	6.8 (-3)	9.2 (-4)	1.4 (-4)	6.1 (-5)	5.8 (-5)
1/8	2.2 (-8)	9.4 (-5)	4.4 (-3)	7.1 (-4)	7.8 (-5)	1.4 (-5)	7.4 (-6)

$S_h = V_h = F_1^h$

h \ ε	1	0.1	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-10}
1/4	3.7 (-7)	7.5 (-4)	1.4 (-3)	1.8 (-5)	2.0 (-6)	1.4 (-6)	1.4 (-6)
1/8	2.5 (-8)	4.9 (-5)	1.4 (-3)	2.8 (-5)	3.4 (-6)	1.2 (-7)	9.9 (-8)

$S_h = P_2^h,\ V_h = F_1^h$

h \ ε	1	0.1	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-10}
1/4	9.0 (-10)	5.1 (-6)	1.2 (-3)	1.6 (-5)	1.5 (-6)	1.8 (-8)	1.0 (-9)
1/8	1.0 (-11)	3.5 (-7)	6.6 (-4)	3.9 (-5)	3.1 (-6)	3.3 (-8)	3.8 (-11)

$S_h = P_3^h,\ V_h = F_2^h$

h \ ε	1	0.1	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-10}
1/4	3.9 (-13)	1.0 (-6)	7.1 (-4)	1.3 (-5)	1.9 (-6)	1.3 (-8)	3.3 (-11)
1/8	4.1 (-13)	6.4 (-9)	1.8 (-4)	4.5 (-5)	3.9 (-6)	2.9 (-8)	3.4 (-11)

$S_h = P_4^h,\ V_h = F_3^h$

TABLE 3

The orders of the error for several Petrov–Galerkin methods

	S_h	V_h	order of the error at mesh points: $h \ll \varepsilon$	order of the error at mesh points: $\varepsilon \ll h$	Remark
1	P_{k+1}^h	P_{k+1}^h	h^{2k+2}	1	
2	E_k^h	E_k^h	h^{2k+2}	1	
3	F_k^h	F_k^h	h^{2k+2}	$\varepsilon + h^{2k}$	*)
4	$E_{k-1}^h + F_{k-1}^h$	$E_{k-1}^h + F_{k-1}^h$	h^{2k+2}	$\varepsilon^2 + h^{2k-2}$	*)
5	E_k^h	P_{k+1}^h	h^{2k+2}	$\varepsilon + h^{k+1}$	
6	E_k^h	F_k^h	h^{2k+2}	$\varepsilon^2 + h^{2k+1}$	
7	P_{k+1}^h	F_k^h	h^{2k+2}	$\frac{\varepsilon^2}{h} + h^{2k+1}$	*)

*) the order of the h-term for $\varepsilon \ll h^2$ might be slightly pessimistic. For details see table 4.

TABLE 4

Experimentally determined orders of convergence for $\varepsilon \ll h$.

	S_h	V_h	k=1	k=2	k=3
3	F_k	F_k	$\varepsilon+h^3$	$\varepsilon+h^{4.5}$	*)
4	$E^h_{k-1}+F^h_{k-1}$	$E^h_{k-1}+F^h_{k-1}$	no experiment	ε^2+h^3	ε^2+h^4
7	P_{k+1}	F_k	$\frac{\varepsilon^2}{h}+h^{3.5}$	$\frac{\varepsilon^2}{h}+h^5$	*)

*) the error was too small to determine the rate of convergence.

In the case $p(x)h \ll \varepsilon$, table 3 shows an error which is much smaller than the theoretical error in table 1. This error of order $\mathcal{O}(h^{2k+2})$ is easily understood since, in the case where $p(x)h \ll \varepsilon$, the trial spaces E^h_k, F^h_k and $E^h_{k-1}+F^h_{k-1}$ differ only slightly from the piecewise polynomial spaces P^h_{k+1} and, in fact, have nearly the same approximation properties for smooth functions.

In the more interesting case $\varepsilon \ll p(x)h$, the theoretical error bounds consist of ε-dependent and h-dependent parts. To perceive in an error of the form $\varepsilon^p + h^q$ the orders of both parameters separately, we have performed experiments both with $\varepsilon^p \ll h^q$ and with $\varepsilon^p \gg h^q$.

(4c) *Numerical instability for* $S^h = V^h = E^h_k$

In table 3 we notice that the 2nd method ($S^h = V^h = E^h_k$) does not follow the theoretically derived error bound. This is due to the fact that for small $\alpha_i h_i$ the basis functions on the subinterval (x_{i-1}, x_i) are almost linearly dependent and a jump occurs at the right-hand side of the subinterval. Hence, the 2nd up to the (k+1)st row in the element stiffness matrix are almost linearly dependent.

This causes cancellation of digits when the assembled stiffness-matrix is solved. After static condensation we obtain a tridiagonal matrix with elements on the subdiagonal of order ε, as follows

$$\begin{matrix} 1 & 0 & & & \\ \varepsilon & 1 & 1 & & \\ & \varepsilon & 1 & 1 & \\ & & \varepsilon & 1 & 1 \\ & & & 0 & 1 \end{matrix} \; .$$

This shows that, in first approximation, the reduced equation is solved with the right-hand boundary condition, whereas the original problem is, again in first approximation, the solution of the reduced problem with the left-hand boundary condition. Thus, it is easily seen that for a problem of which the solution contains a singular part, the numerical approximation has an error of order unity.

(4d) *Conclusions*

The theoretical and the experimental results agree as far as the ε-dependence of the error is concerned, except for the case where $S^h = V^h = E_k^h$. For this particular combination of the solution and test space the Petrov-Galerkin method is numerically unstable.

Concerning the order of the error with respect to h, the theoretical bounds given in theorem 1 and 2 seem to be pessimistic. In some cases (the methods 3, 5 and 6 in table 3) it seems that the order of the error can be increased by one.

Taking into account that the order of the error of the best approximation in E_k^h is $O(\varepsilon+h^{k+1})$ in energy norm, one might expect that the Galerkin approximation has the same error in energy norm. This implies that the approximation at the mesh-points would improve by one order in h.

This better estimate would agree with the experiments. Moreover, it would yield a reasonable error estimate for the case k = 0. However, how the better estimate can be proved remains an open question.

REFERENCES

Abrahamsson, L.R., Keller, H.B. and Kreiss, H.O. (1974). "Difference approximations for singular perturbations of systems of ordinary differential equations". *Numer. Math.*, 22 (1974) 367-391.

Douglas, J. and Dupont, T. (1974). "Galerkin approximations for the two-point boundary problem using continuous piecewise polynomial spaces". *Numer. Math.* 22 (1974) 99-109.

Eckhaus, W. (1973). *Matched asymptotic expansions and singular perturbations*. North-Holland Publ. Comp., Amsterdam, 1973.

De Groen, P.P.N. (1978). "A finite element method with a large mesh-width for a stiff two-point boundary value problem", *to appear*.

Hemker, P.W. (1977). *A numerical study of stiff two-point boundary problems*. MC-Tract 80, Mathematical Centre, Amsterdam.

Il'in, A.M. (1969). "Differencing scheme for a differential equation with a small parameter affecting the highest derivative". *Math. Notes Acad. Sc. USSR* 6 (1969) 596-602.

O'Malley, R.E. (1974). *Introduction to singular perturbations*. Academic Press, New York, 1974.

Van Veldhuizen (1979). "Higher order schemes of positive type for Singular Perturbation Problems". *These proceedings*.