Discrete Approximation of Singularly Perturbed Parabolic PDEs with a Discontinuous Initial Condition

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Abstract

In this paper a Dirichlet problem for a singularly perturbed parabolic partial differential equation with a discontinuous initial condition is considered. We construct a new difference scheme which converges in the ℓ^∞-norm over the whole domain, uniformly in the small parameter. In this scheme the coefficients are fitted to the discontinuity. For two typical numerical examples we compare numerical results obtained for a classical and for the new difference scheme. We compare the theoretical results and the results in practice, and we show that the new method is more accurate and converges uniformly in ϵ indeed.

1 Introduction

We study parabolic PDEs that are singularly perturbed and have a discontinuous initial condition. The discontinuity introduces a singularity at $t = 0$ and ℓ^∞-convergence problems appear in its neighbourhood for the classical numerical schemes.

Outside a fixed neighbourhood of the discontinuity classical difference schemes converge for each fixed value of the small parameter, but they do not converge uniformly for this parameter in any neighbourhood of the interior layer [1, 2]. Therefore, it is interesting to study ℓ^∞-convergence both for regular and for singularly perturbed PDEs with discontinuous initial functions. In the singularly perturbed case it is important to see if convergence can be uniform in the small parameter.

As was shown in [1, 2, 3] for boundary value problems with parabolic layers, no scheme exists that converges uniformly on a uniform grid for the general problem. However, for the restricted class of problems with constant coefficients the present method has this favourable property, and -moreover- numerical examples show that the method is more accurate in practical situations.

2 Problem formulation

We consider the Dirichlet BVP for the following singularly perturbed equation of parabolic type (the subscript number for the symbol L indicates the equation in which it is defined)

$$\begin{array}{ll} L_{(1)}u(x,t) = f(x,t), & (x,t) \in G = (-1,1)\times(0,T], \\ u(x,t) = \phi(x,t), & (x,t) \in S = \overline{G}\backslash G, \end{array} \qquad (1a)$$

where

$$L_{(1)} \equiv \epsilon^2 \frac{\partial^2}{\partial x^2} - p\frac{\partial}{\partial t}. \qquad (1b)$$

The parameter ϵ can take any value $\epsilon \in (0,1]$. The source $f(x,t)$ is a sufficiently smooth functions on $\overline{G}$ and $p > 0$. The boundary function $\phi(x,t)$ has a discontinuity of the first kind on the set $S^* = \{(0,0)\}$. For simplicity S^* consists of a single point only. Outside S^* the function $\phi(x,t)$ is sufficiently smooth on S.

For each fixed value of the parameter ϵ, the solution of problem (1) is continuous and sufficiently smooth for $t > 0$. The discontinuity appears only at the point $(0,0)$. The derivatives exist and are sufficiently smooth in $\overline{G}$, outside a neighbourhood of S^*. They only increase, without bound, in the vicinity of S^*. When the parameter tends to zero, an interior layer appears and the derivatives with respect to x increase without bound, also in the neighbourhood of all the interior layer.

We introduce the standard function

$$w_0(x,t) = w_0(x,t;p) = \frac{1}{2}v\left(\frac{x}{2\epsilon}\sqrt{\frac{p}{t}}\right), \qquad (2)$$

where

$$v(\xi) = \frac{2}{\sqrt{\pi}}\int_0^\xi \exp(-\alpha^2)\, d\alpha$$

is related with the error function. The function $w_0(x,t)$ is the solution of the homogeneous equation

$$\left\{\epsilon^2 \frac{\partial^2}{\partial x^2} - p\frac{\partial}{\partial t}\right\}\, u(x,t) = 0, \quad (x,t)\in G. \qquad (3)$$

This function is piecewise constant on S at $t = 0$ and has a discontinuity of the first kind in S^*, with the jump $[w_0(0,0)] = 1$, defined by $[v(x,t)] = \lim_{s\searrow 0} v(x+s,t) - \lim_{s\nearrow 0} v(x+s,t)$.

To see the effect of the singularity, we split the solution of (1) in a part with and a part without the singular behaviour. Suppose

$$W(x,t) = [\phi(0,0)]\, w_0(x,t;p), \quad (x,t)\in\overline{G}^{(*)} = \overline{G}\backslash S^*,$$

then the function $W(x,t)$ is continuous, except at S^* where $[W(x,t)] = [u(x,t)] = [\phi(x,t)]$. We now write the solution of problem (1) as a sum

$$u(x,t) = W(x,t) + U(x,t), \quad (x,t)\in\overline{G}^{(*)}, \qquad (4)$$

where $U(x,t)$ is the solution of the problem

$$\begin{array}{ll} L_{(1)}U(x,t) = f(x,t), & (x,t)\in G, \\ U(x,t) = \phi(x,t) - W(x,t), & (x,t)\in S. \end{array} \qquad (5)$$

On S the function $U(x,t)$ is continuous and piecewise smooth. For simplicity we suppose that $U(x,t)$ is sufficiently smooth on the boundary of G, and that a compatibility condition is satisfied at the corner points. We are interested in the solution of problem (1) in the neighbourhood of the point of discontinuity and the generated interior layer. Therefore, we suppose that the boundary conditions at $x = \pm 1$ are such that no boundary layers appear.

3 An ϵ-uniformly convergent scheme

On the set $\overline{G}$ we introduce the rectangular grid $\overline{G}_h = \{\omega \times \omega_0\} \cap \overline{G}$. Here ω and ω_0 are uniform grids on the segments $[-1,1]$ and $[0,T]$ respectively, and (possibly $S_h^* = \emptyset$)

$$G_h = G \cap \overline{G}_h; \quad S_h = S \cap \overline{G}_h; \quad S_h^* = S^* \cap \overline{G}_h.$$

For the numerical approximation of (1) we may use classical difference approximations. If we take the usual implicit central difference scheme we have

$$\begin{array}{ll} \Lambda_{(6)}z(x,t) = f(x,t), & (x,t)\in G_h, \\ z(x,t) = \phi(x,t), & (x,t)\in S_h, \end{array} \qquad (6a)$$

where

$$\Lambda_{(6)} \equiv \delta_{x\bar{x}} - p\,\delta_{\bar{t}}, \qquad (6b)$$

with $\delta_{\bar{t}}z(x,t)$ and $\delta_{x\bar{x}}z(x,t)$ the usual first and second difference of $z(x,t)$ on the uniform grids ω_0 and ω respectively; the bar denotes the backward difference; the time step is denoted by τ and the space step by h. It is well known that the operator $\Lambda_{(6)}$ is monotone. It implies that the maximum principle holds for (6).

For the approximation of equation (1) we now introduce a new, specially fitted scheme

$$\begin{array}{ll} \Lambda_{(7)}z(x,t) = f(x,t), & (x,t)\in G_h, \\ z(x,t) = \phi(x,t), & (x,t)\in S_h, \end{array} \qquad (7a)$$

where

$$\Lambda_{(7)} \equiv \epsilon^2\gamma(x,t)\delta_{x\bar{x}} - p\,\delta_{\bar{t}}\,. \tag{7b}$$

Here $\gamma(x,t)$ is a fitting coefficient, which could be chosen such that the solution of the homogeneous differential equation satisfies the homogeneous difference equation

$$\Lambda_{(7)}w_0(x,t) \equiv \left\{\epsilon^2\gamma(x,t)\delta_{x\bar{x}} - p\,\delta_{\bar{t}}\right\} w_0(x,t) = 0, \quad (x,t)\in G_h\,. \tag{8}$$

As a better alternative we may select $\gamma(x,t)$ such that (8) is satisfied by $u(x,t) = w_0(x,t) + u_0(x,t)$, where w_0 is the singular solution and u_0 is some smooth solution of the homogeneous equation

$$L_{(1)}u(x,t) = 0, \quad (x,t)\in G\,. \tag{9}$$

This leads to the following expression for γ:

$$\gamma(x,t) = \frac{p}{\epsilon^2}\frac{\delta_{\bar{t}}v(x,t)}{\delta_{x\bar{x}}v(x,t)}, \quad (x,t)\in G_h\,. \tag{10}$$

for any point (x,t) where $\delta_{x\bar{x}}v(x,t) \neq 0$. Here

$$v(x,t) = w_0(x,t,p) + \eta u_0(x,t), \quad (x,t)\in G_h, \tag{11}$$

with η some suitable parameter. We notice, for $\eta = 0$, that $\delta_{x\bar{x}}v(x,t)$ and $\delta_{\bar{t}}v(x,t)$ can be very small because of the exponentially small derivatives of $w_0(x,t;p)$ for large $x/(\epsilon\sqrt{t})$. Therefore, we choose the function ηu_0 such that the differences $\delta_{x\bar{x}}w_0$, $\delta_{x\bar{x}}u_0$, $\delta_{\bar{t}}w_0$ and $\delta_{\bar{t}}u_0$ all have the same sign, for $(x,t)\in G_h$. For this we can take

$$u_0(x,t) = -px^3 - 6\epsilon^2 xt, \quad (x,t)\in\overline{G}, \tag{12}$$

and $\eta > 0$. Then

$$\gamma(x,t) = \gamma(x,t;\eta) = \frac{p}{\epsilon^2}\cdot\frac{\delta_{\bar{t}}w_0(x,t) + \eta\delta_{\bar{t}}u_0(x,t)}{\delta_{x\bar{x}}w_0(x,t) + \eta\delta_{x\bar{x}}u_0(x,t)}, \quad x \neq 0\,. \tag{13}$$

We notice that both $\delta_{x\bar{x}}v = 0$ and $\delta_{\bar{t}}v = 0$ at $x = 0$. For definiteness we set $\gamma(0,t) = 1$. Now we define our new difference scheme as (7), where $\gamma(x,t)$ is determined by (13). Using the techniques from [1, 2]. the following theorem is proved.

Theorem For $h,\tau \to 0$, if the relation $\tau^{3/2}/h \to 0$ is satisfied, then, the solution of the difference scheme (7)-(13) converges to the solution of problem (1) uniformly in ϵ:

$$\|u - z\|_{\ell^\infty,\overline{G}_h} = \max_{\overline{G}_h}|u(x,t) - z(x,t)| \leq M(h^\nu + \tau^\nu + \tau^{3/2}/h)\,,$$

with $\nu\in[0,1]$. For $h = \tau^{\frac{3}{2(1+\nu)}}$ we have the estimate

$$\|u - z\|_{\ell^\infty,\overline{G}_h} \leq M(h^{2\nu_1} + \tau^{\nu_1}), \quad (x,t)\in\overline{G}_h\,, \tag{14}$$

for any $\nu_1\in(\frac{1}{2},\frac{3}{4})$.

Theory shows that the new scheme converges uniformly in ϵ on $\overline{G}$, but no indication is given about the value of the order constant M in (14) and, moreover, the order of convergence is rather small. It is possible that the error $\|z - u\|_{\infty,\overline{G}_h}$ is relatively large for any reasonable value of h or τ. This might reduce the practical use of our new scheme. To decide on the practical value of the new scheme numerical experiments give the final answer.

The numerical behaviour of the new scheme

Numerical results that will be reported elsewhere show that classical scheme (6) does not converge in the ℓ^∞-norm for any fixed ϵ. On a domain excluding the region $0 < t < t_0$ the scheme converges for a fixed ϵ, but it does not converge uniformly in ϵ.

Here we are interested in the behaviour of the new scheme when applied to the problem (1) with $f \equiv 0$, where the function $u(x,t)$ is the sum of a smooth and a singular part

$$u(x,t) = u_1(x,t) + w_0(x,t), \quad (x,t)\in\overline{G}^{(*)}\,. \tag{15}$$

For clarity we examine the behaviour for the two components separately. For the smooth part we take

$$u_1(x,t) = -(x+0.5)^2 - 2\epsilon^2 t\,.$$

A number of numerical experiments was made and led to the following conclusions: (i) for both $w_0(x,t)$ and $u_1(x,t)$ the numerical scheme (7)-(13) converges in the ℓ^∞-norm for a fixed ϵ; (ii) as the error $E(h,\tau,\epsilon) = \|z - u\|_{\ell^\infty,\overline{G}_h}$ decreases with decreasing ϵ we observe ϵ-uniform convergence, both for $w_0(x,t)$ and for $u_1(x,t)$.

We can examine the experimental order of convergence for the new scheme. We assume the 'generalised' order of convergence ν to appear in the expression for the error as

$$E(h,\tau) = \max_\epsilon \max_{\overline{G}_h^{(*)}} |u(x,t,\epsilon) - z(x,t,\epsilon,h,\tau)| \leq M(h^2 + \tau)^\nu\,.$$

From this assumption we determine the experimental 'generalised' order in the point (h,τ) as

$$\nu(h,\tau,\epsilon) = (\ln E(h,\tau) - \ln E(h/2,\tau/2))/\ln 4\,.$$

We introduce the *experimental order of convergence*, for a given value of the parameter, as

$$\nu(\epsilon) = \max_{h,\tau}\nu(h,\tau,\epsilon)\,. \tag{16}$$

and the experimental ϵ-uniform order as $\nu = \max_\epsilon \nu(\epsilon)$. Similarly the experimental ϵ-uniform order in the point (h,τ) is $\nu(h,\tau) = \max_\epsilon \nu(h,\tau,\epsilon)$. In Table 1 we show the observed orders of convergence for the singular and the smooth parts of the solution separately.

For the order of convergence numerical experiments led to the following observations: (iii) for $w_0(x,t)$ and $u_1(x,t)$ the experimental order of convergence for the new numerical scheme is approximately 0.413 and 0.450 respectively; (iv) the order of convergence increases with decreasing h and τ, and for $h \leq 1/16$ and $\tau \leq 1/40$ the orders of

	$u = w_0$				$u = u_1$			
τ	h				h			
	1/8	1/16	1/32	1/64	1/8	1/16	1/32	1/64
1/10	.583	.736	.789	.795	.544	.479	.454	.450
1/40	.677	.850	.949	.992	.631	.653	.640	.650
1/160	.413	.530	.828	1.012	.681	.782	.792	.818

Table 1: The experimental order of convergence $\nu(h,\tau)$.

convergence for $w_0(x,t)$ and $u_1(x,t)$ are apparently not less than 0.5. This means that in practice

$$\max_{\overline{G}_h}|u(x,t) - z(x,t)| \leq M(h + \tau^{1/2})$$

for $h < 1/16$ and $\tau < 0.1$.

These results are essentially better than for the classical scheme. For the classical difference scheme convergence is absent: for a fixed value of ϵ this scheme doesn't converge in the ℓ^∞-norm in the neighbourhood of the discontinuity, and away from the discontinuity it does not converge ϵ-uniformly in the neighbourhood of the interior layer.

4 References

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