THE INCOMPLETE LU-DECOMPOSITION AS A RELAXATION METHOD IN MULTI-GRID ALGORITHMS

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ABSTRACT

We consider relaxation methods for use with Multi-Grid (MG-) algorithms to solve a sparse linear system

$$ A_{i,j} x_{i,j} = f_{i,j}, \quad j > 0, $$

which is supposed to be the discretization of a continuous boundary value problem $A x = f$. With the same equation coarser discretizations are related:

$$ A_k x_k = f_k, \quad k = 0, 1, \ldots, \ell-1. $$

We first give a brief exposition of the framework of MG-methods. Next, we describe the incomplete LU-decomposition as a relaxation method and in the third section we compare different variants of it. The conclusion is in favour of the ILU variant.

THE MULTI-GRTID FRAMEWORK

Considering MG methods in which each iteration step consists of $\gamma$ coarse grid correction steps, preceded by $p$ and followed by $q$ relaxation sweeps, we see that in each MG iteration step the residual is multiplied by the operator (cf. [2])

$$ M_{k-1}^{\text{MLA}} = M_k^{\text{TLA}} + (M_k^{\text{REL}})^p A_k^{-1} A_{k-1}^{-1} (M_k^{\text{TLA}})^p, \quad k = 1, 2, \ldots, \ell. $$

where

- $M_k^{\text{TLA}} = 0$
- $M_k^{\text{MLA}} = (M_k^{\text{REL}})^p (I - A_k^{-1} A_{k-1}^{-1}) (M_k^{\text{REL}})^p$
- $P_k$ denotes the prolongation from level $k-1$ to level $k$
- $R_k$ denotes the restriction from level $k$ to level $k-1$
- $A_k^{-1}$ denotes the $k-1$ level discretization of the operator $A$, e.g.

$$ A_k^{-1} = R_k^{-1} A_k P_k^{-1}. $$

$M_k^{\text{MLA}}$ is the operator on level $k$ by which the residual is multiplied in one relaxation sweep (see next section).

Communication between finer and coarser grids takes place via the prolongations and restrictions. By the spectral decomposition of a grid function

$$ u_h(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{ij\omega} \hat{u}_h(\omega) \, d\omega $$

where

$$ \hat{u}_h(\omega) = \frac{1}{\sqrt{2\pi}} \sum_{j \in \mathbb{Z}} e^{-ij\omega} u_h(\omega) $$

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denotes the spectrum of the gridfunction $u_h$, defined on $[-\pi/h, \pi/h]^n$, we see that only low frequency components of a gridfunction can be represented on coarse grids. The relation between the spectra of a gridfunction, its prolongation and its restriction are given by

$$(H = qh)$$

$$\hat{\mathbf{R}}_h(\omega) = \left(\frac{2\pi}{H}\right)^n \sum_{0 \leq p < q} \hat{a}_h(\omega+2\pi p/H) \hat{u}_h(\omega+2\pi p/H),$$

$$\hat{\mathbf{P}}_h(\omega) = \left(\frac{2\pi}{H}\right)^n \hat{b}_h(\omega) \hat{u}_h(\omega);$$

where $a_h$ and $b_h$ are grid functions that characterize the particular prolongations and restrictions (cf.[3]).

With the choice $A_{k-1} = R_h P$, we find the relation

$$(I - \hat{P} R)(I - A_k P A_{k-1}^{-1} R) = (I - A_k P A_{k-1}^{-1} R)$$

for any prolongation $P$. Hence, taking e.g. Shannon's interpolation for $P$, we see that - in the residual - coarse grid corrections annihilate all frequencies that can be represented on the coarser grid. Similar results hold for other reasonable choices of $A_{k-1}$.

From this it is clear that an efficient relaxation method in an MG-algorithm should damp those high frequencies that cannot be represented on the coarser grid (in the 2-D case $n=2$) and with mesh doubling ($H=qh=2h$) this is the shadowed portion of fig. 1). This behaviour is analyzed by considering $M(\omega)$, the spectrum of the operator $M^{REL}$. Following BRANDT [1], we define the smoothing rate

$$\mu = \sup_{\omega \in [-\pi/h, \pi/h]^n} |\hat{M}(\omega)|.$$  

For any linear difference operator $A_k$ with constant coefficients and each relaxation method this $\mu$ can easily be determined numerically. E.g. it is well known that for the usual 5-point discretization of Poisson's equation ($n=2$, $q=2$) and GS-relaxation we have $\mu = 0.5$.

![Fig. 1. Frequency region](image)

We consider similar relaxation methods in order to find methods that take less work per iteration sweep and smaller values of $\mu$.  

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INCOMPLETE LU-DECOMPOSITION AS A RELAXATION METHOD

By $B_k$ we denote the approximate inverse in a stationary relaxation method for the solution of $A_k x_k = f_k$. Then $M_k^{\text{REL}} = I - A_k B_k$ is the operator by which the residual is multiplied in each relaxation sweep. E.g. with Jacobi-iteration $B_k = (\text{diag}(A_k))^{-1}$ and with forward- or backward Gauss-Seidel iteration

$$ B_k = (\text{lower triag}(A_k))^{-1} \quad \text{resp.} \quad B_k = (\text{upper triag}(A_k))^{-1}. $$

We are interested in incomplete LU-decomposition, where for some $D, L$ and $U$ (which are diagonal, lower and upper triangular respectively) we take

$$ B_k = (LD^{-1}U)^{-1} $$

and we may write

$$ A_k = LD^{-1}U + R. $$

Although the treatment of incomplete LU-decomposition can be given for the general case, here we confine ourselves to infinite Toeplitz-matrices $A_k, L, D$ and $U$ of the form: $D = wI$,

$$ (*) \quad A_k = \begin{pmatrix} C & G & E \\ B & H & \end{pmatrix}, \quad L = \begin{pmatrix} a & b \\ c & \end{pmatrix}, \quad U = \begin{pmatrix} \alpha & \beta \\ \gamma & \end{pmatrix}. $$

It can be shown that the analysis for this case is representative for the local behaviour in the interior of the domain of the PDE. Multiplication of $LD^{-1}U$ and identification of the sub-and superdiagonals yields

$$ C = H, \quad B = B, \quad G = G, \quad \gamma = E, \quad a = \alpha = w^{-1}, $$

and $R$ has the form

$$ R = \begin{pmatrix} r_m & 0 & ru & 0 \\ 0 & ru & 0 & \end{pmatrix}, \quad ru = BE/a, $$

$$ r_l = HG/a, \quad rm = a - C + (GD+HE)/a. $$

Here one free parameter $a$ is left. If no corrections are admitted to the main diagonal: $rm = 0$ and $a$ is prescribed by

(ILU) \quad a = \frac{1}{2} \left( C \pm \sqrt{C^2 - 4(GD+HE)} \right). $$

To get minimal $ru$ and $r_l$ the sign which yields maximal absolute value is used. Other choices for $a$ are possible:

(SGS) \quad a = C,

or the 1-st order approximation

(MILU) \quad a = C - (GD+HE)/C.

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Remark.

Notice that the choice \( a = C \) yields a method which is equivalent with symmetric Gauss-Seidel relaxation. In this case namely \( A_K = L + U - D \) and we see

\[
\hat{M}^{\text{REL}} = I - A_K(LD^{-1}U)^{-1} = (I - A_KU^{-1})(I - A_KL^{-1}),
\]

i.e. one GS-forward step followed by one GS-backward step is identical with this particular incomplete LU-iteration step. Symmetric GS-iteration (SGS) is one (simple) form of incomplete LU-iteration.

THE EFFICIENCY OF INCOMPLETE LU-ITERATION

To study the efficiency of the incomplete LU-relaxation, first we compute the smoothing rate \( \mu(a) \) for the usual 5-point discretization of Poisson's equation, depending on the parameter \( a \). The maximum of

\[
\hat{M}(\omega) = \frac{(a^2 - 4a + 2) + 2 \cos(\omega_1 - \omega_2)}{(a^2 + 2) - 2a(\cos \omega_1 - \cos \omega_2) + 2 \cos(\omega_1 - \omega_2)}
\]

on the shaded area of fig. 1 can be computed analytically and is \((a>2)\)

\[
\mu(a) = \frac{8 \pm \sqrt{A^2 + 12}}{8 - 2Aa \pm (A+4a)\sqrt{A^2 + 12}}, \quad \text{where } A = a^2 - 4a + 2,
\]

and is attained at \( \omega = (\arccos\left(\frac{8 + \sqrt{A^2 + 12}}{A^2 + 16}\right), \frac{\pi}{2}) \).

A graph of this function is given in fig. 2. For our three particular choices of \( a \) we find:

- SGS: \( a = 4 \), \( \mu(a) = 0.2500 \), \( \omega = (\arccos(4/5), \pi/2) \);
- MILU: \( a = 3.5 \), \( \mu(a) = 0.1649 \), \( \omega = (0.555, \pi/2) \);
- ILU: \( a = 2 + \sqrt{2} \), \( \mu(a) = 0.2035 \), \( \omega = (\pi/3, \pi/2) \).

The minimal value of \( \mu(a) \) is 0.1607 and is attained for \( a = 3.510 \) which is remarkably close to our choice MILU.

The amount of work in each iteration step is \( 5N \) for the solution and \( 3N \) for the computation of the residual (\( 2N \) in the case ILU, where \( rm = 0 \)). Summarizing, the efficiency of the different methods is

- SGS: \( \mu = 0.25 \), work \( 8N \), efficiency \( \varphi_0.25 = 0.84 \);
- MILU: \( \mu = 0.16 \), work \( 8N \), efficiency \( \varphi_0.16 = 0.80 \);
- ILU: \( \mu = 0.20 \), work \( 7N \), efficiency \( \varphi_0.20 = 0.79 \).

For the Poisson equation we conclude that, although MILU has optimal \( \mu(a) \), ILU has slightly better efficiency because of the cheaper computation of the residual.
Fig. 2 The smoothing rate $\mu(a)$.

Beside Poisson's equation we also considered other model equations viz. the convection-diffusion equation

$$\varepsilon \Delta \phi + \cos(\alpha) \frac{\partial \phi}{\partial x} + \sin(\alpha) \frac{\partial \phi}{\partial y} = f,$$

discretized with Il'in's method and the anisotrope Poisson equation

$$\varepsilon \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f.$$

For the convection diffusion we see from (*) that, asymptotically for $\varepsilon \to 0$, either H or E and either B or G vanish. Hence, asymptotically the methods SGS, MILU and ILU coincide. Moreover, for $0 < \alpha < \frac{\pi}{2}$, mod($\pi$), asymptotically $\mu \to 0$, i.e. the system is solved exactly by only one iteration sweep. However, for a convection direction $\alpha$ with $-\frac{\pi}{2} \leq \alpha \leq 0$, mod($\pi$), $\mu$ takes a positive value. For different values of $\varepsilon$ and $\alpha$, these $\mu$ are given in table 1. Taking into account the number of operations, we see again that, although $\mu_{\text{MILU}} \leq \mu_{\text{SILU}} \leq \mu_{\text{SGS}}$, ILU is in most cases more efficient than MILU.

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<th>$-\pi/4$</th>
<th>$-\pi/8$</th>
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<th>$\pi/2$</th>
<th>$3\pi/8$</th>
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<td>0.405</td>
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**Table 1.** The smoothing rate $\mu$ for the convection diffusion equation with Il'in's discretization.
<table>
<thead>
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<th>$\varepsilon = 1$</th>
<th>$\varepsilon = 0.5$</th>
<th>$\varepsilon = 0.1$</th>
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<td>SGS</td>
<td>0.250</td>
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<td>MILU</td>
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<td>0.190</td>
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<td>ILU</td>
<td>0.204</td>
<td>0.250</td>
<td>0.477</td>
<td>0.768</td>
</tr>
</tbody>
</table>

Table 2. $\nu$ for the anisotope Poisson equation.

For the anisotope Poisson equation (table 2) we see that in the three cases $\nu + 1$ as $\varepsilon \to 0$. However, MILU and ILU have smaller smoothing rates and for small $\varepsilon$ we even have $\nu_{\text{ILU}} < \nu_{\text{MILU}}$.

We conclude that also in this case the ILU-decomposition is the best choice in our class of incomplete LU-decomposition relaxation methods and it is significantly more efficient than symmetric Gauss-Seidel relaxation.

**REFERENCES**


