# $\varepsilon$-Uniform Schemes with High-Order Time-Accuracy for Parabolic Singular Perturbation Problems. The Neumann Problem *

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**Abstract**

We study the discrete approximation of a Neumann problem on an interval for a singularly perturbed parabolic PDE. For a boundary value problem we construct the special piecewise-uniform mesh on which the difference scheme based on the classical finite difference approximation converges $\varepsilon$-uniformly with the order $\mathcal{O}(N^{-2}\ln^2 N + K^{-1})$, where $N+1$ and $K+1$ are the number of the nodes at the space and the time meshes, respectively. On using such schemes we construct the schemes of the high order accuracy with respect to the time. To obtain the better approximation, we use auxiliary discrete problems on the same time-mesh to correct the difference approximations. To validate the theoretical results, some numerical results for the new schemes are presented.

## 1 Introduction

Under smooth data of singularly perturbed boundary value problems for parabolic equations without convection terms, the order of $\varepsilon$-uniform convergence for the special scheme that was studied (i.e. [1]–[6]) is $\mathcal{O}(N^{-2}\ln^2 N + K^{-1})$, where $N$ and $K$ denote, respectively, the number of intervals in the space and time discretisation. For this scheme the amount of computational work is primarily determined by the time discretisation, which is of first order accuracy only. The improvement of the order accuracy in time, maintaining $\varepsilon$-uniform convergence, by means of a defect correction technique was studied in [7] for a Dirichlet problem and it was achieved without essentially increasing the amount of computational work.

Here we use the similar method for a Neumann problem and also we obtain the higher order of accuracy with respect to the time variable and the second (up to logarithmic multiplier) order accuracy in space.

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## 2 The class of boundary value problems studied

1. On the domain $G = D \times (0,T]$, $D = (0,1)$ with the boundary $S = \overline{G} \setminus G$ we consider the singularly perturbed parabolic equation with Neumann boundary conditions:

$$L_{(2.1)} u(x,t) \equiv \varepsilon^2 \frac{\partial}{\partial x}\left( a(x,t) \frac{\partial}{\partial x} u(x,t) \right) - c(x,t) u(x,t) - \tag{2.1a}$$

$$-p(x,t) \frac{\partial}{\partial t} u(x,t) = f(x,t), \quad (x,t) \in G, \quad \varepsilon \in (0,1],$$

$$l_{(2.1)} u(x,t) \equiv \varepsilon \frac{\partial}{\partial n} u(x,t) = \psi(x,t), \quad (x,t) \in S_1, \tag{2.1b}$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S_0.$$

Here $S = S_0 \cup S_1$, $S_1 = \{(x,t): \ x = 0 \text{ and } x = 1, \ 0 < t \le T\}$, $S_0 = \{(x,t): \ x \in [0,1], \ t = 0\}$, $\partial/\partial n$ is the derivative with respect to the normal to $S_1$ external to set $G$. In (2.1) $a(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$, $(x,t) \in \overline{G}$, and $\varphi(x,t)$, $(x,t) \in S_0$, $\psi(x,t)$, $(x,t) \in S_1$ are sufficiently smooth and bounded functions

$$0 < a_0 \le a(x,t), \quad 0 < p_0 \le p(x,t), \quad c(x,t) \ge 0, \quad (x,t) \in \overline{G}.$$

When the parameter $\varepsilon$ tends to zero, in a neighbourhood of the lateral boundary $S_1$ layers appear in the solution, which are described by an equation of parabolic type.

2. We will suppose that the compartibility conditions which ensure sufficient smoothness of the problem solution are satisfied on the set $S_0 \cap \overline{S_1}$. Then for the solution of the problem and its components from the representation

$$u(x,t) = U(x,t) + W(x,t), \quad (x,t) \in \overline{G},$$

where $U(x,t)$, $W(x,t)$ are the regular and singular parts, the following estimates hold [1]

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \, \partial t^{k_0}} u(x,t) \right| \le M \, \varepsilon^{-k}, \quad \left| \frac{\partial^{k+k_0}}{\partial x^k \, \partial t^{k_0}} U(x,t) \right| \le M, \tag{2.2}$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \, \partial t^{k_0}} W(x,t) \right| \le M \, \varepsilon^{-k} \exp(-m\varepsilon^{-1} r(x,\Gamma)\,),$$

$$(x,t) \in \overline{G}, \quad k + 2k_0 \le 2n+4,$$

where $r(x,\Gamma)$ is the distance between the point $x \in \overline{D}$ and the set $\Gamma = \overline{D} \setminus D$.

## 3 The finite difference schemes

1. To solve problem (2.1) we first consider a classical finite difference method. On the set $\overline{G}$ we introduce the rectangular mesh

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{3.1}$$

where $\overline{\omega}$ is the (possibly) non-uniform mesh of nodal points, $x^i$, in $[0,1]$, $\overline{\omega}_0$ is a uniform mesh on the interval $[0,T]$; $N$ and $K$ are the numbers of intervals in the grids $\overline{\omega}$ and $\overline{\omega}_0$

[1] Here and below we denote by $M$ (or $m$) sufficiently large (or small) positive constants which do not depend on the value of parameter $\varepsilon$ or on the difference operators.

respectively. We define $\tau = T/K$, $h^i = x^{i+1} - x^i$, $h = \max_i h^i$, $h \le M/N$, $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$.

For problem (2.1) we use the difference scheme

$$\Lambda_{(3.2)} z(x,t) = f(x,t), \quad (x,t) \in G_h, \tag{3.2a}$$

$$\lambda_{(3.2)} z(x,t) = \psi^h(x,t), \quad (x,t) \in S_{1h}, \qquad z(x,t) = \varphi\,(x,t), \quad (x,t) \in S_{0h}. \tag{3.2b}$$

Here

$$\Lambda_{(3.2)} z(x,t) \equiv \varepsilon^2 \delta_{\widehat{x}} \left( a^h(x,t) \delta_{\overline{x}} z(x,t) \right) - c(x,t) z(x,t) - p(x,t) \delta_{\overline{t}} z(x,t),$$
$$\lambda_{(3.2)} z(x,t) \equiv -\varepsilon \delta_x z(x,t) + 2^{-1} \varepsilon^{-1} (x^{i+1} - x^i)(a^{+h}(x,t))^{-1} [c(x,t)z(x,t) + p(x,t)\delta_{\overline{t}} z(x,t)],$$
$$\psi^h(x,t) = a(x,t)(a^{+h}(x,t))^{-1} \psi(x,t) - 2^{-1}\varepsilon^{-1}(x^{i+1} - x^i)(a^{+h}(x,t))^{-1} f(x,t), \quad x = x^i = 0;$$
$$\lambda_{(3.2)} z(x,t) \equiv \ \varepsilon \delta_{\overline{x}} z(x,t) + 2^{-1}\varepsilon^{-1}(x^i - x^{i-1})(a^h(x,t))^{-1}[c(x,t)z(x,t) + p(x,t)\delta_{\overline{t}} z(x,t)],$$
$$\psi^h(x,t) = a(x,t)(a^h(x,t))^{-1}\psi(x,t) - 2^{-1}\varepsilon^{-1}(x^i - x^{i-1})(a^h(x,t))^{-1} f(x,t), \quad x = x^i = 1;$$
$$\delta_{\widehat{x}} \left( a^h(x^i,t)\ \delta_{\overline{x}} z(x^i,t) \right) = 2(x^{i+1} - x^{i-1})^{-1} \left( a^{+h}(x^i,t) \delta_x z(x^i,t) - a^h(x^i,t) \delta_{\overline{x}} z(x^i,t) \right),$$
$$a^h(x^i,t) = a\left((x^{i-1} + x^i)/2, t\right), \quad a^{+h}(x^i,t) = a^h(x^{i+1},t) = a\left((x^i + x^{i+1})/2, t\right),$$

$\delta_x z(x,t)$ and $\delta_{\overline{x}} z(x,t)$, $\delta_{\overline{t}} z(x,t)$ are the forward and backward differences, and the difference operator $\delta_{\widehat{x}}(a^h(x,t)\delta_{\overline{x}} z(x,t))$ is an approximation of the operator $\frac{\partial}{\partial x}(a(x,t)\frac{\partial}{\partial x} u(x,t))$ on the non-uniform mesh.

Taking into account estimates of the derivatives we find that the solution of the difference scheme (3.2), (3.1) converges for a fixed value of the parameter $\varepsilon$:

$$|\ u(x,t) - z(x,t)\ | \le M(\varepsilon^{-1} N^{-1} + \tau), \quad (x,t) \in \overline{G}_h. \tag{3.3}$$

2. We construct the scheme convergent $\varepsilon$-uniformly. On $\overline{G}$ we introduce the mesh

$$\overline{G}_h^{\,*} = \overline{\omega}^*(\sigma) \times \overline{\omega}_0\ , \tag{3.4}$$

where $\overline{\omega}_0 = \overline{\omega}_{0(3.1)}$ and $\overline{\omega}^* = \overline{\omega}^*(\sigma)$ is a special *piecewise* uniform mesh depending on the parameter $\sigma \in I\!\!R$, $\sigma = \sigma_{(3.4)}(\varepsilon, N) = \min[\,d/4,\ m\varepsilon \ln N\,]$. The mesh $\overline{\omega}^*(\sigma)$ is constructed as follows. The interval $[\,0,1\,]$ is divided in three parts $[\,0,\sigma\,]$, $[\,\sigma, 1-\sigma\,]$, $[\,1-\sigma, 1\,]$, $0 < \sigma \le 1/4$. In each part we use a uniform mesh, with $N/2$ subintervals in $[\,\sigma, 1-\sigma\,]$ and with $N/4$ subintervals in each interval $[\,0,\sigma\,]$ and $[\,1-\sigma, 1\,]$.

**Theorem 3.1** *Let the estimate* (2.2) *hold for the solution of* (2.1). *Then the solution of* (3.2), (3.4) *converges $\varepsilon$-uniformly to the solution of* (2.1) *with the error bounds*

$$|u(x,t) - z(x,t)| \le M(N^{-2}\ln^2 N + \tau), \quad (x,t) \in \overline{G}_h^*. \tag{3.5}$$

# 4 Improved time-accuracy. A scheme based on defect correction

Here we adapt the discrete method based on defect correction [7], which ensures $\varepsilon$-uniform convergence of the approximate solution to the solution of (2.1) with an order of time-accuracy higher than two. But for the Neumann problem it is necessary also to made the correction of the difference derivative $\delta_{\overline{t}} z(x,t)$ in the boundary condition.

We denote by $\delta_{k\bar t} z(x,t)$ the backward difference of order $k$:

$$\delta_{k\bar t}\, z(x,t) = \left(\delta_{k-1\,\bar t}\, z(x,t) - \delta_{k-1\,\bar t}\, z(x,t-\tau)\right)/\tau, \quad t \ge k\tau, \quad k \ge 1;$$
$$\delta_{0\bar t}\, z(x,t) = z(x,t), \quad (x,t) \in \overline{G}_h.$$

1. Let us construct modified difference schemes of the second order accuracy in $\tau$ for the boundary value problem (2.1). On the mesh $\overline{G}_{h(3.4)}$ we consider the "usual" finite difference scheme (3.2), writing

$$\Lambda_{(3.2)} z^{(1)}(x,t = f(x,t), \quad (x,t) \in G_h, \tag{4.1}$$
$$\lambda_{(3.2)} z^{(1)}(x,t) = \psi^h(x,t), \ \ (x,t) \in S_{1h}, \quad z^{(1)}(x,t) = \varphi(x,t), \ \ (x,t) \in S_{0h}.$$

Then for problem (2.1) we come to the "improved" difference scheme:

$$\Lambda_{(3.2)} z^{(2)}(x,t) = f(x,t) + \begin{cases} p(x,t) 2^{-1}\,\tau\, \frac{\partial^2}{\partial t^2} u(x,0), & t = \tau, \\ p(x,t) 2^{-1}\,\tau\, \delta_{2\,\bar t} z^{(1)}(x,t), & t \ge 2\tau, \quad (x,t) \in G_h, \end{cases} \tag{4.2}$$

$$\lambda_{(3.2)} z^{(2)}(x,t) = \psi^h(x,t), \ \ (x,t) \in S_{1h}, \quad z^{(2)}(x,t) = \varphi(x,t), \ \ (x,t) \in S_{0h}.$$

Here $z^{(1)}(x,t)$ is the solution of the discrete problem (4.1), (3.4), and the derivative $(\partial^2/\partial t^2) u(x,0)$ is obtained from the equation (2.1a). We shall call $z^{(2)}(x,t)$ the solution of difference scheme (4.2), (4.1), (3.4) (or shortly, (4.2), (3.4)).

**Theorem 4.1** *Let the condition $\varphi(x,0) = 0$, $x \in \overline{D}$ holds and assume in equation* (2.1) *that $a \in H^{(\alpha+2n+1)}(\overline{G})$, $c$, $p$, $f \in H^{(\alpha+2n)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n \ge 0$ and let $u \in H^{(\alpha+2n)}$ for $n = 1$. Then for the solution of difference scheme* (4.2), (3.4) *the following estimate holds*

$$\left| u(x,t) - z^{(2)}(x,t) \right| \le M \left[\, N^{-2} \ln^2 N + \tau^2 \,\right], \quad (x,t) \in \overline{G}_h. \tag{4.3}$$

3. Analogously we construct a difference scheme with third order accuracy in $\tau$. For problem (2.1) on the mesh $\overline{G}_{h(3.4)}$ we consider the difference scheme

$$\Lambda_{(3.2)} z^{(3)}(x,t) = f(x,t) +$$
$$+ \begin{cases} p(x,t)(C_{11}\tau \frac{\partial^2}{\partial t^2} u(x,0) + C_{12}\tau^2 \frac{\partial^3}{\partial t^3} u(x,0)), & t = \tau, \\ p(x,t)(C_{21}\tau \frac{\partial^2}{\partial t^2} u(x,0) + C_{22}\tau^2 \frac{\partial^3}{\partial t^3} u(x,0)), & t = 2\tau, \\ p(x,t)(C_{31}\tau \delta_{2\,\bar t} z^{(2)}(x,t) + C_{32}\tau^2 \delta_{3\,\bar t} z^{(1)}(x,t)), & t \ge 3\tau, \quad (x,t) \in G_h, \end{cases} \tag{4.4}$$

$$\lambda_{(3.2)} z^{(3)}(x,t) = \psi^h(x,t) -$$
$$- \begin{cases} 4^{-1}\varepsilon^{-1}(x^{i+1} - x^i)\tau (a^{+h}(x,t))^{-1} p(x,t) \frac{\partial^2}{\partial t^2} u(x,0), & x = x^i = 0, \\ 4^{-1}\varepsilon^{-1}(x^i - x^{i-1})\tau (a^h(x,t))^{-1} p(x,t) \frac{\partial^2}{\partial t^2} u(x,0), & x = x^i = 1, \quad t = \tau, \end{cases}$$

$$\lambda_{(3.2)} z^{(3)}(x,t) = \psi^h(x,t) -$$
$$- \begin{cases} 4^{-1}\varepsilon^{-1}(x^{i+1} - x^i)\tau (a^{+h}(x,t))^{-1} p(x,t) \delta_{2\bar t} z^{(1)}(x,t), & x = x^i = 0, \\ 4^{-1}\varepsilon^{-1}(x^i - x^{i-1})\tau (a^h(x,t))^{-1} p(x,t) \delta_{2\bar t} z^{(1)}(x,t), & x = x^i = 1, \quad t \ge 2\tau, \end{cases}$$
$$(x,t) \in S_{1h},$$

$$z^{(3)}(x,t) = \varphi(x,t), \quad (x,t) \in S_{0h}.$$

Here the derivatives $(\partial^2/\partial t^2)u(x,0)$, $(\partial^3/\partial t^3)u(x,0)$ are obtained from (2.1a), the coefficients $C_{ij}$ are determined by $C_{11} = C_{21} = C_{31} = 1/2$, $C_{12} = C_{32} = 1/3$, $C_{22} = 5/6$.

We shall call $z^{(3)}(x,t)$ the solution of the difference scheme (4.4), (3.4).

Under the condition $\varphi(x,0) = 0$, $f(x,0) = 0$, $x \in \overline{D}$, the following estimate holds for the solution of difference scheme (4.4), (3.4)

$$\left| u(x,t) - z^{(3)}(x,t) \right| \le M \left[ N^{-2} \ln^2 N + \tau^3 \right], \quad (x,t) \in \overline{G}_h. \tag{4.5}$$

In a similar way we can construct difference schemes with $\varepsilon$-uniform order convergence $\mathcal{O}(N^{-2} \ln^2 N + \tau^n)$, $n > 3$, i.e. an arbitrary high order of time-accuracy.

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# Appendix

## 5 Numerical results for the time-accurate schemes

1. We consider the singularly perturbed boundary value problem with the Neumann condition. The solution of the problem in the half-strip,

$$L_{(5.1)}V(x,t) \equiv \left\{ \varepsilon^2 \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \right\} V(x,t) = 0, \quad 0 < x < \infty, \ 0 < t \le T, \tag{5.1}$$

$$\varepsilon \frac{\partial}{\partial x} V(0,t) = -(128/35)\, \pi^{(-1/2)}\, t^{7/2}, \quad 0 < t \le T, \quad V(x,0) = 0, \quad 0 \le x < \infty,$$

is given by

$$V(x,t)=\operatorname{erfc}\left(\frac{x}{2\varepsilon\sqrt{t}}\right)\left(\frac{x^8}{1680\varepsilon^8}+\frac{x^6}{30\varepsilon^6}\,t+\frac{x^4}{2\varepsilon^4}\,t^2+\frac{2x^2}{\varepsilon^2}\,t^3+t^4\right)- \tag{5.2}$$

$$-\frac{1}{\sqrt{\pi}}\exp\left(\frac{-x^2}{4\varepsilon^2 t}\right)\left(\frac{x^7}{840\varepsilon^7}\,t^{1/2}+\frac{9x^5}{140\varepsilon^5}\,t^{3/2}+\frac{37x^3}{42\varepsilon^3}\,t^{5/2}+\frac{93x}{35\varepsilon}\,t^{7/2}\right).$$

We consider the model problem

$$L_{(5.1)}u(x,t)=0,\quad (x,t)\in G, \tag{5.3}$$

$$\varepsilon\frac{\partial}{\partial x}u(x,t)=\varepsilon\frac{\partial}{\partial x}V_{(5.2)}(x,t),\quad (x,t)\in S_1,\quad x=0,$$

$$u(x,t)=V_{(5.2)}(x,t),\quad (x,t)\in S,\quad x\neq 0.$$

Then the function $V_{(5.2)}(x,t)$ is the solution of problem (5.3).

| $\varepsilon$ | $K$ \ $N$ | 8 | 32 | 128 | 512 | 2048 |
|---|---|---|---|---|---|---|
| 1 | 8 | 1.01(-1) | 1.08(-1) | 1.08(-1) | 1.08(-1) | 1.08(-1) |
| | 32 | 2.15(-2) | 2.73(-2) | 2.78(-2) | 2.78(-2) | 2.78(-2) |
| | 128 | 2.73(-3) | 6.53(-3) | 6.96(-3) | 6.99(-3) | 6.99(-3) |
| | 512 | 5.94(-3) | 1.35(-3) | 1.72(-3) | 1.75(-3) | 1.75(-3) |
| | 2048 | 7.26(-3) | 1.72(-4) | 4.09(-4) | 4.36(-4) | 4.37(-4) |
| $2^{-2}$ | 8 | 6.98(-2) | 1.20(-1) | 1.26(-1) | 1.27(-1) | 1.27(-1) |
| | 32 | 7.56(-2) | 2.51(-2) | 3.09(-2) | 3.13(-2) | 3.14(-2) |
| | 128 | 1.01(-1) | 3.99(-3) | 7.36(-3) | 7.79(-3) | 7.82(-3) |
| | 512 | 1.07(-1) | 5.78(-3) | 1.56(-3) | 1.92(-3) | 1.95(-3) |
| | 2048 | 1.09(-1) | 7.25(-3) | 2.49(-4) | 4.60(-4) | 4.86(-4) |
| $2^{-4}$ | 8 | 1.87(-1) | 7.76(-2) | 1.20(-1) | 1.26(-1) | 1.27(-1) |
| | 32 | 2.91(-1) | 5.16(-2) | 2.51(-2) | 3.09(-2) | 3.13(-2) |
| | 128 | 3.17(-1) | 7.64(-2) | 3.99(-3) | 7.36(-3) | 7.79(-3) |
| | 512 | 3.23(-1) | 8.26(-2) | 5.78(-3) | 1.56(-3) | 1.92(-3) |
| | 2048 | 3.25(-1) | 8.42(-2) | 7.25(-3) | 2.49(-4) | 4.60(-4) |
| $2^{-6}$ | 8 | 1.87(-1) | 7.76(-2) | 1.16(-1) | 1.26(-1) | 1.27(-1) |
| | 32 | 2.91(-1) | 5.16(-2) | 2.30(-2) | 3.02(-2) | 3.13(-2) |
| | 128 | 3.17(-1) | 7.64(-2) | 3.46(-3) | 6.79(-3) | 7.71(-3) |
| | 512 | 3.23(-1) | 8.26(-2) | 9.38(-3) | 1.22(-3) | 1.85(-3) |
| | 2048 | 3.25(-1) | 8.42(-2) | 1.09(-2) | 6.97(-4) | 3.98(-4) |

Table 1: Table of errors $E(N,K,\varepsilon)$ for scheme (4.1), (5.5)

$E(N,K,\varepsilon)$ is defined by

$$E(N,K,\varepsilon)=\max_{(x,t)\in\overline{G}_h}|\,z(x,t)-u^*(x,t)\,|, \tag{5.4}$$

where $z(x,t)=z^{(1)}_{(4.1,5.5,5.6)}(x,t)$, $u^*(x,t)=V_{(5.2)}(x,t)$, $\overline{G}_h\ =\ \overline{G}^{(*)}_{h(5.6)}$.

2. Strictly saying, problem (5.3) is the problem with mixed boundary conditions, i.e., the Neumann and the Dirichlet conditions at the left and the right boundaries respectively. The solution has a boundary layer, and at the point $x=1$, $V(x,t)$ is exponentially small in $\varepsilon^{-1}$ for

$\varepsilon \to 0$. Therefore actually problem (5.3) is the problem with the Neumann condition. We use for the approximation of problem (5.3) the schemes (for Dirichlet condition at $x = 1$), which are formed for $x < 1$ by the mesh equations (4.1), (3.4); (4.2), (3.4) and (4.4), (3.4) and for $x = 1$ by the following mesh equations

$$z^{(k)}(x,t) = V_{(5.2)}(x,t), \quad (x,t) \in S, \quad x = 1. \tag{5.5}$$

As the solution of boundary value problem (5.3) has a boundary layer at the left side, for its solution we use the locally condensed mesh

$$\overline{G}_h^{(*)} = \overline{\omega}^{(*)} \times \overline{\omega}_0 \ , \tag{5.6}$$

where $\overline{\omega}^{(*)} = \overline{\omega}^{(*)}(\sigma)$ is a special mesh, condensed in the neighbourhood of the left end of the interval $[\,0,1\,]$; $\sigma$ is the parameter depending on $\varepsilon$ and $N$. The mesh $\overline{\omega}^{(*)}(\sigma)$ is a *piecewise* constant mesh with constant steps $h_{(1)}$ and $h_{(2)}$ on the intervals $[\,0,\sigma\,]$ and $[\,\sigma,1\,]$, $h_{(1)} = \sigma\,(N/2)^{-1}$, $h_{(2)} = (1-\sigma)\,(N/2)^{-1}$. We take $\sigma = \min[\,1/2,\ 2\varepsilon \ln N\,]$.

According to the theory, the difference schemes (4.1), (5.5), (5.6); (4.2), (5.5), (5.6) and (4.4), (5.5), (5.6) converge respectively with order 1, 2 and 3 with respect to $\tau$.

To demonstrate this effect numerically, we solve problem (5.3), using these schemes for various values of $N$, $K$ and $\varepsilon$.

3. Numerical results for the above model problem are given in Tables 1–3.

| $\varepsilon$ | $K$ \ $N$ | 8 | 32 | 128 | 512 | 2048 |
|---|---|---|---|---|---|---|
| 1 | 8 | 2.94(-2) | 2.01(-2) | 1.63(-2) | 1.53(-2) | 1.51(-2) |
| | 32 | 1.11(-3) | 2.13(-3) | 1.38(-3) | 1.11(-3) | 1.03(-3) |
| | 128 | 5.98(-3) | 7.75(-5) | 1.38(-4) | 8.84(-5) | 7.06(-5) |
| | 512 | 7.27(-3) | 3.76(-4) | 4.91(-6) | 8.71(-6) | 5.55(-6) |
| | 2048 | 7.59(-3) | 4.58(-4) | 2.35(-5) | 2.83(-7) | 5.94(-7) |
| $2^{-2}$ | 8 | 1.75(-2) | 2.92(-2) | 1.85(-2) | 1.45(-2) | 1.34(-2) |
| | 32 | 8.49(-2) | 1.41(-3) | 2.05(-3) | 1.24(-3) | 9.47(-4) |
| | 128 | 1.03(-1) | 5.96(-3) | 9.27(-3) | 1.32(-4) | 7.86(-5) |
| | 512 | 1.08(-1) | 7.31(-3) | 3.75(-4) | 5.87(-6) | 8.30(-6) |
| | 2048 | 1.09(-1) | 7.64(-3) | 4.60(-4) | 2.34(-5) | 4.45(-7) |
| $2^{-4}$ | 8 | 1.95(-1) | 1.98(-2) | 2.92(-2) | 1.85(-2) | 1.45(-2) |
| | 32 | 2.93(-1) | 6.28(-2) | 1.41(-3) | 2.05(-3) | 1.24(-3) |
| | 128 | 3.17(-1) | 7.92(-2) | 5.96(-3) | 9.27(-5) | 1.32(-4) |
| | 512 | 3.23(-1) | 8.33(-2) | 7.31(-3) | 3.75(-4) | 5.87(-6) |
| | 2048 | 3.25(-1) | 8.43(-2) | 7.64(-3) | 4.60(-4) | 2.34(-5) |
| $2^{-6}$ | 8 | 1.95(-1) | 1.98(-2) | 3.06(-2) | 2.12(-2) | 1.58(-2) |
| | 32 | 2.93(-1) | 6.28(-2) | 2.39(-3) | 2.30(-3) | 1.54(-3) |
| | 128 | 3.17(-1) | 7.92(-2) | 9.21(-3) | 4.53(-4) | 1.50(-4) |
| | 512 | 3.23(-1) | 8.33(-2) | 1.08(-2) | 1.01(-3) | 5.52(-5) |
| | 2048 | 3.25(-1) | 8.43(-2) | 1.12(-2) | 1.14(-3) | 9.75(-5) |

Table 2: Table of Errors $E(N,K,\varepsilon)$ for scheme (4.2), (5.5)

$E(N,K,\varepsilon)$ is defined by (5.4), where $z(x,t) = z^{(2)}_{(4.2,5.5,5.6)}(x,t)$, $u^*(x,t) = V_{(5.2)}(x,t)$, $\overline{G}_h = \overline{G}^{(*)}_{h(5.6)}$.

| $N$ | | 8 | 32 | 128 | 512 | 2048 |
|---|---|---|---|---|---|---|
| $\varepsilon$ | $K$ | | | | | |
| 1 | 8 | 2.14(-3) | 2.70(-3) | 2.52(-3) | 2.44(-3) | 2.42(-3) |
| | 32 | 6.31(-3) | 3.29(-4) | 2.70(-5) | 3.81(-5) | 3.79(-5) |
| | 128 | 7.38(-3) | 4.61(-4) | 2.78(-5) | 1.09(-6) | 6.06(-7) |
| | 512 | 7.62(-3) | 4.80(-4) | 3.00(-5) | 1.87(-6) | 1.37(-7) |
| | 2048 | 7.68(-3) | 4.84(-4) | 3.03(-5) | 1.91(-6) | 1.50(-7) |
| $2^{-2}$ | 8 | 5.32(-2) | 2.04(-3) | 2.49(-3) | 2.45(-3) | 2.41(-3) |
| | 32 | 9.52(-2) | 6.35(-3) | 3.50(-4) | 1.78(-5) | 2.92(-5) |
| | 128 | 1.06(-1) | 7.41(-3) | 4.64(-4) | 2.83(-5) | 1.29(-6) |
| | 512 | 1.08(-1) | 7.66(-3) | 4.82(-4) | 3.01(-5) | 1.88(-6) |
| | 2048 | 1.09(-1) | 7.72(-3) | 4.87(-4) | 3.04(-5) | 1.91(-6) |
| $2^{-4}$ | 8 | 2.19(-1) | 3.83(-2) | 2.04(-3) | 2.49(-3) | 2.45(-3) |
| | 32 | 2.99(-1) | 7.32(-2) | 6.35(-3) | 3.50(-4) | 1.78(-5) |
| | 128 | 3.19(-1) | 8.18(-2) | 7.41(-3) | 4.64(-4) | 2.83(-5) |
| | 512 | 3.24(-1) | 8.40(-2) | 7.66(-3) | 4.82(-4) | 3.01(-5) |
| | 2048 | 3.25(-1) | 8.45(-2) | 7.72(-3) | 4.87(-4) | 3.04(-5) |
| $2^{-6}$ | 8 | 2.19(-1) | 3.83(-2) | 2.01(-3) | 2.39(-3) | 2.49(-3) |
| | 32 | 2.99(-1) | 7.32(-2) | 9.37(-3) | 9.15(-4) | 5.12(-5) |
| | 128 | 3.19(-1) | 8.18(-2) | 1.09(-2) | 1.13(-3) | 1.05(-4) |
| | 512 | 3.24(-1) | 8.40(-2) | 1.12(-2) | 1.17(-3) | 1.10(-4) |
| | 2048 | 3.25(-1) | 8.45(-2) | 1.13(-2) | 1.18(-3) | 1.11(-4) |

Table 3: Table of Errors $E(N, K, \varepsilon)$ for scheme (4.4), (5.5)

$E(N, K, \varepsilon)$ is defined by (5.4), where $z(x,t) = z^{(3)}_{(4.4,5.5,5.6)}(x,t)$, $u^*(x,t) = V_{(5.2)}(x,t)$, $\overline{G}_h = \overline{G}^{(*)}_{h(5.6)}$.

Thus, the numerical results confirm the theoretical results and demonstrate the efficiency of the defect correction.