Parallel Methods Based on a Defect-Correction Technique for Parabolic Singularly Perturbed Problems *

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Abstract

The discrete approximation of a Dirichlet problem on an interval for a singularly perturbed parabolic PDE is studied. A small parameter ε multiplies the highest-order derivative. For small values of the parameter, boundary layers appear that give rise to difficulties when classical discretization methods are applied.

For well-known special difference schemes the order of convergence is one and two, up to a small logarithmic factor, with respect to the time and space variables, respectively. To obtain ε-uniform convergence, we used a grid with nodes that are condensed in the neighbourhood of the boundary layers. To obtain the better approximation in time, we used auxiliary discrete problems on the same time-grid to correct the difference approximations. It allows us to receive an arbitrarily large order of convergence in time if the solution is sufficiently smooth. In this paper we develop effective parallel algorithms to solve the discrete equations based on defect correction. To construct such algorithms, we use a modified Schwartz alternating process.

1. Introduction

Special ε-uniformly convergent difference schemes for singularly perturbed boundary value problems for parabolic equations are well developed, see, e.g., [1]–[3], [5]–[7]. If the problem data are sufficiently smooth, for the parabolic equations without convection terms, the order of ε-uniform convergence for the scheme that was studied is $\mathcal{O}(N^{-2}\ln^2 N + N_0^{-1})$, where N and N_0 denote, respectively, the number of intervals in the space and time discretization.

In [1, 2] we have developed an algorithm based on the defect correction principle which achieves a high order of accuracy with respect to the time variable and the second-order accuracy in space. In [7] parallel computational methods were introduced that allowed us to accelerate the numerical solution of singularly perturbed boundary value problems for parabolic reaction-diffusion equations. It is attractively to use both technique as defect correction as parallel algorithm as well.

In the present paper we develop a new parallel computational method to solve the system of grid equations arising when the defect correction technique is used for an approximations of the boundary value problem. By this way, we can achieve a high order of accuracy for the time variable, maintaining ε-uniform convergence high-order accuracy in time, as well as a high efficiency of the algorithms due to parallel computations. It should be noted that this parallel method does not require iterations at each time level.

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2. Problem formulation

On the domain $G = (0,1) \times (0,T]$, with the boundary $S = \overline{G} \setminus G$ we consider the boundary value problem for singularly perturbed parabolic equation

$$
\begin{aligned}
L_{(2.1)}u(x,t) &\equiv \left\{\varepsilon^2 a(x,t)\frac{\partial^2}{\partial x^2} - c(x,t) - p(x,t)\frac{\partial}{\partial t}\right\} u(x,t) = f(x,t), \quad (x,t) \in G,\\
u(x,t) &= \varphi(x,t), \quad (x,t) \in S.
\end{aligned}
\tag{2.1}
$$

Here $S = S_0 \cup S_1$, $S_1 = \{(x,t):\ x = 0 \text{ or } x = 1,\ 0 < t \le T\}$, $S_0 = \{(x,t):\ x \in [0,1],\ t = 0\}$; $a(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$, $(x,t) \in \overline{G}$, and $\varphi(x,t)$, $(x,t) \in S$ are sufficiently smooth and bounded functions, ε takes any value from $(0,1]$.

When the parameter ε tends to zero in (2.1), in the neighbourhood of the lateral boundary layers of parabolic type appear in the solution.

For problem (2.1) we are to construct a numerical method that has a higher order of accuracy with respect to the time variable and, in addition, admits parallel computations for the solution of the difference equations.

3. Difference scheme on special mesh

To solve problem (2.1) we first consider a classical finite difference method. On the set $\overline{G}$ we introduce the rectangular grid

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{3.1}$$

where $\overline{\omega}$ is the (possibly) non-uniform grid of nodal points, x^i, in $[0,1]$, $\overline{\omega}_0$ is a uniform grid on the interval $[0,T]$; N and N_0 are the numbers of intervals in the grids $\overline{\omega}$ and $\overline{\omega}_0$ respectively. We define $\tau = T/N_0$, $h^i = x^{i+1} - x^i$, $h = \max_i h^i$, $h \le M/N$, $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$, M is sufficiently large positive constant, independent on ε.

For problem (2.1) we use the difference scheme [4]

$$\Lambda_{(3.2)}z(x,t) = f(x,t), \quad (x,t) \in G_h, \quad z(x,t) = \varphi(x,t), \quad (x,t) \in S_h, \tag{3.2}$$

where $\Lambda_{(3.2)}z(x,t) \equiv \{\varepsilon^2 a(x,t)\delta_{\overline{x}\widehat{x}} - c(x,t) - p(x,t)\delta_{\overline{t}}\}\, z(x,t)$,

$$\delta_{\overline{x}\widehat{x}}z(x^i,t) = 2(h^{i-1} + h^i)^{-1}[\delta_x z(x^i,t) - \delta_{\overline{x}}z(x^i,t)],$$

$\delta_x z(x,t)$ and $\delta_{\overline{x}}z(x,t)$, $\delta_{\overline{t}}z(x,t)$ are the forward and backward differences.

To provide an ε-uniform convergence of the difference scheme we use a special mesh, condensed in the neighbourhood of boundary layers [1, 2, 5, 6]:

$$\overline{G}_h^{\,*} = \overline{\omega}^*(\sigma) \times \overline{\omega}_0\,. \tag{3.3}$$

Here $\overline{\omega}_0 = \overline{\omega}_{0(3.1)}$, $\overline{\omega}^* = \overline{\omega}^*(\sigma)$ is a special *piecewise* uniform mesh, $\sigma = \sigma_{(3.3)}(\varepsilon,N) = \min[\,d/4,\ m\varepsilon\ln N\,]$, where $m = m_{(3.3)}$ is an arbitrary positive number. The mesh $\overline{\omega}^*(\sigma)$ is constructed as follows. The interval $[0,1]$ is divided in three parts $[0,\sigma]$, $[\sigma, 1-\sigma]$, $[1-\sigma, 1]$, $0 < \sigma \le 1/4$. In each part we use a uniform grid, with $N/2$ subintervals in $[\sigma, 1-\sigma]$ and with $N/4$ subintervals in each interval $[0,\sigma]$ and $[1-\sigma,1]$.

We assume that at the corner points $S_0 \cap \overline{S}_1$ the following conditions hold

$$
\begin{aligned}
&\frac{\partial^k}{\partial x^k}\varphi(x,t) = \frac{\partial^{k_0}}{\partial t^{k_0}}\varphi(x,t) = 0, \quad && k + 2k_0 \le [\,\alpha\,] + 2n,\\
&\frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}}f(x,t) = 0, && k + 2k_0 \le [\,\alpha\,] + 2n - 2,
\end{aligned}
\tag{3.4}
$$

where $[\,\alpha\,]$ is the integer part of a number α, $\alpha > 0$, $n \ge 0$ is an integer number. We also suppose that $[\,\alpha\,] + 2n \ge 2$. We denote by $H^{(\alpha)}(\overline{G}) = H^{\alpha,\alpha/2}(\overline{G})$ the Hölder space, where α is an arbitrary positive number.

4. Difference schemes based on the Schwartz method

For problem (2.1) we describe the Schwartz method that admits parallel computations on $P \geq 1$ solvers [7].

4.1. Suppose, the set of subdomains

$$D^k, \quad k = 1, \ldots, \quad K \tag{4.1}$$

with boundaries Γ^k, $\Gamma^k = \Gamma(D^k) = \overline{D}^k \setminus D^k$, forms the covering of the set D: $D = \bigcup_{k=1}^{K} D^k$. Let each subdomain $D^k_{(4.1)}$ be multiply connected and be formed by the union of P, $P \geq 1$ nonoverlapping domains (some of them may be empty):

$$D^k = \bigcup_{p=1}^{P} D^k_p, \quad k = 1, \ldots, K, \quad \overline{D}^k_i \cap \overline{D}^k_j = \emptyset, \ i \neq j. \tag{4.2}$$

We set

$$G^k_p = D^k_p \times (0, T], \ G^k_p(t^n) = D^k_p \times (t^{n-1}, t^n], \ t^n \in \overline{\omega}_0, \ p = 1, \ldots, P, \ k = 1, \ldots, K. \tag{4.3}$$

We denote by $D^{[k]}$ the union of the subdomains $D^1, \ldots, D^K$ which do not have the set D^k: $D^{[k]} = \bigcup_{i=1,\, i \neq k}^{K} D^i$. We denote the minimal width of the overlapping of the sets D^k and $D^{[k]}$ by δ^k. Let δ denote the least value of δ^k, $k = 1, \ldots, K$, i.e.

$$\min_{k,\, x^1,\, x^2} \rho(x^1, x^2) = \delta, \quad x^1 \in \overline{D}^k, \quad x^2 \in \overline{D}^{[k]}, \quad x^1, x^2 \notin \{\, D^k \cap D^{[k]} \,\}, \tag{4.4}$$

$k = 1, \ldots, K$, $\rho(x^1, x^2)$ is the distance between the points x^1, $x^2 \in \overline{D}$.

Suppose that

$$\delta = \delta_{(4.4)}(\varepsilon) > 0, \quad \varepsilon \in (0, 1], \quad \inf_{\varepsilon \in (0,1]} [\, \varepsilon^{-1} \delta_{(4.4)}(\varepsilon) \,] > 0. \tag{4.5}$$

We find the function $u(x,t)$ by the solution of the problems

$$L_{(2.1)} u_p^{\frac{k}{K}}(x,t) = f(x,t), \quad (x,t) \in G^k_p(t^n), \tag{4.6}$$

$$u_p^{\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} \overline{u}(x,t;t^{n-1}), & k = 1, \\ u^{\frac{k-1}{K}}(x,t), & k \geq 2 \end{array} \right\}, \quad (x,t) \in S^k_p(t^n), \quad p = 1, \ldots, P$$

for $(x,t) \in \overline{G}^k_p(t^n)$; where

$$u^{\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} u_p^{\frac{k}{K}}(x,t), \ (x,t) \in \overline{G}^k_p(t^n), & p = 1, \ldots, P, \\ \left. \begin{array}{ll} \overline{u}(x,t;t^{n-1}), & k = 1, \\ u^{\frac{k-1}{K}}(x,t), & k \geq 2 \end{array} \right\}, & (x,t) \in \overline{G}(t^n) \setminus \bigcup_{p=1}^{P} \overline{G}^k_p(t^n) \end{array} \right\}$$

for $(x,t) \in \overline{G}(t^n)$; $\quad k = 1, \ldots, K$;

$$u(x,t) = u^{\frac{K}{K}}(x,t), \quad (x,t) \in \overline{G}(t^n), \quad t^n \in \overline{\omega}_0, \ n = 0, 1, \ldots, N_0 - 1.$$

Here $\overline{u}(x,t;t^{n-1}) = \varphi(x,t)$, $(x,t) \in S(t^{n-1}) \bigcap S$, $\overline{u}(x,t;t^{n-1}) = u(x,t^{n-1})$, $(x,t) \in G(t^n)$, $\overline{G}(t^n) = \overline{D} \times [t^{n-1}, t^n]$.

4.2. We give the difference scheme that approximates differential scheme (4.6), (4.) with P parallel solvers. We introduce the rectangular grids on each set $\overline{G}_p^k$:

$$\overline{G}_{ph}^k = \overline{G}_p^k \bigcap \overline{G}_h, \quad \text{where } \overline{G}_h = \overline{G}_{h(3.1)} \text{ or } \overline{G}_h = \overline{G}^*_{h(3.3)}. \tag{4.7}$$

We find the functions $z^{\frac{k}{K}}(x,t)$ at the strip $\overline{G}_h(t^n)$ by the solution of such problems

$$\Lambda_{(3.2)}(z_p^{\frac{k}{K}}(x,t)) = f(x,t), \quad (x,t) \in G_{p\,h}^k(t^n), \tag{4.8}$$

$$z_p^{\frac{k}{K}}(x,t) = \left\{ \begin{array}{ll} \overline{z}(x,t;t^{n-1}), & k=1, \\ z^{\frac{k-1}{K}}(x,t), & k \geq 2 \end{array} \right\}, \quad (x,t) \in S_{p\,h}^k(t^n), \quad p=1,\ldots,P,$$

for $(x,t) \in \overline{G}_{p\,h}^k(t^n)$; where

$$z^{\frac{k}{K}}(x,t) = \left\{ \begin{array}{lll} z_p^{\frac{k}{K}}(x,t), & & (x,t) \in \overline{G}_{ph}^k(t^n), \quad p=1,\ldots,P, \\ \overline{z}(x,t;t^{n-1}), & k=1, & \\ z^{\frac{k-1}{K}}(x,t), & k \geq 2 & (x,t) \in \overline{G}(t^n) \setminus \cup_{p=1}^P \overline{G}_p^k(t^n), \end{array} \right.$$

for $(x,t) \in \overline{G}_h(t^n)$; $k=1,\ldots,K$, $t^n \in \overline{\omega}_0$. We define the function $z(x,t)$ at the strip $\overline{G}_h(t^n)$ by the relation

$$z(x,t) = z^{\frac{K}{K}}(x,t), \quad (x,t) \in \overline{G}_h(t^n), \quad t^n \in \overline{\omega}_0. \tag{4.9}$$

Here $\overline{z}(x,t;t^{n-1}) = \varphi(x,t)$, $(x,t) \in S_h(t^{n-1}) \bigcap S$, $\overline{z}(x,t;t^{n-1}) = z(x,t^{n-1}) + \phi(x,t)$, $(x,t) \in G_h(t^n)$; $\overline{G}(t^n)_h = \overline{G}(t^n) \bigcap \overline{G}_h$. If the defect correction is not used, we have $\phi(x,t) = 0$, $(x,t) \in \overline{G}(t^n)$.

We are to find the function $z(x,t)$, $(x,t) \in \overline{G}_h$, i.e., the solution of difference scheme (4.), (4.7).

For these shemes we shall use the operator form

$$Q(z(x,t);,f(.),\phi(.) = 0, \quad (x,t) \in \overline{G}_h(t^n), \tag{4.10}$$

where $\phi(x,t) = 0$.

In the decomposition method of the domain (4.), (4.7) the intermediate problems on the subsets $\overline{D}_{ph}^k = \overline{D}_{p(4.)}^k \bigcap \overline{D}_h$ are solved in parallel for all $p=1,\ldots,P$.

The difference scheme (4.), (4.7) for $P=1$ is the scheme for the sequential computations.

4.3. If condition (4.5) holds, by the comparison theorems we get the estimate

$$| z_{(3.2)}(x,t) - z_{(4.)}(x,t) | \leq M N_0^{-1}, \quad (x,t) \in \overline{G}_h, \tag{4.11}$$

where $z_{(3.2)}(x,t)$ and $z_{(4.)}(x,t)$ are the solutions of difference schemes (3.2), (3.1) and (4.), (4.7), (3.1), respectively.

When using the difference schemes (4.), (4.7), on grids (3.1) or (3.3), under condition (4.5), we obtain the following estimates for the solution of boundary value problem (2.1):

$$| u(x,t) - z_{(4.)}(x,t) | \leq M(\varepsilon^{-1}N^{-1} + \tau), \quad (x,t) \in \overline{G}_{h(3.1)}, \tag{4.12}$$

$$| u(x,t) - z_{(4.)}(x,t) | \leq M(N^{-2}\ln^2 N + \tau), \quad (x,t) \in \overline{G}^*_{h(3.3)}. \tag{4.13}$$

5. Improved time-accuracy

5.1. For the difference scheme (3.2), (3.3) the error in the approximation of the partial derivative $(\partial/\partial t)\, u(x,t)$ is caused by the divided difference $\delta_{\bar t}\, z(x,t)$ and is associated with the truncation error given by the relation

$$\frac{\partial}{\partial t}u(x,t) - \delta_{\bar t}\, u(x,t) = 2^{-1}\,\tau \frac{\partial^2}{\partial t^2}u(x,t) - 6^{-1}\,\tau^2 \frac{\partial^3}{\partial t^3}u(x,t-\vartheta),$$

where $\vartheta \in [0,\tau]$. Therefore we now shall use for the approximation of $(\partial/\partial t)\, u(x,t)$ the expression $\delta_{\bar t}\, u(x,t) + \tau\delta_{\bar t \bar t}\, u(x,t)/2$, where $\delta_{\bar t \bar t}\, u(x,t) \equiv \delta_{t \bar t}\, u(x,t-\tau)$, $\delta_{t \bar t}\, u(x,t)$ is the second central divided difference. We can evaluate a better approximation than (3.2) by defect correction

$$\Lambda_{(3.2)} z^c(x,t) = f(x,t) + p(x,t) 2^{-1} \tau \delta_{\bar t \bar t}\, u(x,t), \tag{5.1}$$

τ is step-size of the grid $\overline{\omega}_0$; $z^c(x,t)$ is the "corrected" solution. Instead of $\delta_{\bar t \bar t}\, u(x,t)$ we shall use $\delta_{\bar t \bar t}\, z(x,t)$, where $z(x,t)$, $(x,t) \in G_{h(3.3)}$ is the solution of the difference scheme (3.2), (3.3). The new solution $z^c(x,t)$ has an accuracy of $\mathcal{O}(\tau^2)$ with respect to the time variable.

We denote by $\delta_{l\,\bar t} z(x,t)$ the backward difference of order l:

$$\delta_{l\bar t}\; z(x,t) = \left(\delta_{l-1\,\bar t}\; z(x,t) - \delta_{l-1\,\bar t}\; z(x,t-\tau)\right)/\tau, \quad t \ge l\tau, \quad l \ge 1;$$

$$\delta_{0\bar t}\; z(x,t) = z(x,t), \quad (x,t) \in \overline{G}_h.$$

5.2. On the grid $\overline{G}_h$ we consider the finite difference scheme (3.2), writing

$$\begin{aligned} \Lambda_{(3.2)} z^{(1)}(x,t) &= f(x,t), \quad (x,t) \in G_h, \\ z^{(1)}(x,t) &= \varphi(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{5.2}$$

When constructing difference schemes of second order accuracy in τ in (5.1), instead of $\delta_{\bar t \bar t}\, u(x,t)$ we use $\delta_{2\,\bar t}\, z^{(1)}(x,t)$, which is the second divided difference of the solution to the discrete problem (5.2), (3.3). Then for the boundary value problem (2.1) we now have the discrete problem:

$$\Lambda_{(3.2)} z^{(2)}(x,t) = f(x,t) + \begin{cases} p(x,t)2^{-1}\,\tau \dfrac{\partial^2}{\partial t^2}u(x,0), & t = \tau, \\ p(x,t)2^{-1}\,\tau\, \delta_{2\,\bar t} z^{(1)}(x,t), & t \ge 2\tau, \quad (x,t) \in G_h, \end{cases} \tag{5.3}$$

$$z^{(2)}(x,t) = \varphi(x,t), \quad (x,t) \in S_h.$$

Here $z^{(1)}(x,t)$ is the solution of the discrete problem (5.2), (3.3), and the derivative $\frac{\partial^2}{\partial t^2}u(x,0)$ is obtained from the equation (2.1). We shall call $z^{(2)}(x,t)$ the solution of difference scheme (5.3), (5.2), (3.3) (or shortly, (5.3), (3.3)).

5.3. For simplicity, we take a homogeneous initial condition:

$$\varphi(x,0) = 0, \quad x \in \overline{D}. \tag{5.4}$$

Under condition (3.4) and (5.4), the following estimate [2] holds for the solution of problem (5.3) (or more strictly (5.3), (5.2))

$$\begin{aligned} \left|u(x,t) - z^{(2)}(x,t)\right| &\le M\,\left[\varepsilon^{-1} N^{-1} + \tau^2\,\right], \qquad (x,t) \in \overline{G}_{h(3.1)}, \\ \left|u(x,t) - z^{(2)}(x,t)\right| &\le M\,\left[\,N^{-2}\ln^2 N + \tau^2\,\right], \quad (x,t) \in \overline{G}_{h(3.3)}. \end{aligned} \tag{5.5}$$

Theorem 5.1 *Let condition* (5.4) *hold and assume in equation* (2.1) *that* $a,\ c,\ p,\ f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n \ge 1$ *and let condition* (3.4) *be satisfied for* $n = 1$. *Then for the solution of difference scheme* (5.3), (3.3) *((*5.3), (3.1)*)* *the estimate* (5.5) *holds.*

In a similar way we could construct difference schemes with an arbitrary high order of accuracy $\mathcal{O}(N^{-2}\ln^2 N + \tau^n)$, $n > 2$.

Numerical results and their analysis, however, without parallel computations, can be seen in [1, 2]. These results demonstrate the efficiency of the defect correction technique in improving the accuracy with respect to the time variable. But the implementation of the schemes in [1, 2] for a finer time-grid may take, generally speaking, a great deal of time. Therefore the parallel methods for the finite difference schemes based on the defect correction technique are required.

6. Parallel method based on defect correction

6.1. On the grid $\overline{G}_h$ we consider the finite difference schemes (4.), (4.7), writing

$$Q(z^{(1)}(x,t);\ f^{(1)}(.),\ \phi^{(1)}(.) = 0, \quad (x,t) \in \overline{G}_h(t^n), \tag{6.1}$$

where $f^{(1)}(x,t) = f(x,t)$, $\phi^{(1)}(x,t) = 0$. To improve accuracy in time we solve the problem

$$Q(z^{(2)}(x,t);\ f^{(2)}(.),\ \phi^{(2)}(.) = 0, \quad (x,t) \in \overline{G}_h(t^n), \tag{6.2}$$

where

$$f^{(2)}(x,t) = f(x,t) + \begin{cases} p(x,t)2^{-1}\,\tau\,\dfrac{\partial^2}{\partial t^2}\,u(x,0), & t = \tau, \\ p(x,t)2^{-1}\,\tau\,\delta_{2\bar{t}}z^{(1)}(x,t), & t \ge 2\tau, \end{cases}$$

$$\phi^{(2)}(x,t) = z^{(1)}(x,t^n) - z^{(1)}(x,t^{n-1}).$$

6.2. For the solution of difference scheme (6.), (4.7) the estimate (5.5) holds (condition (4.5) and the hypothesis of Theorem 5.1 are assumed to be fulfilled).

Theorem 6.1 *Let the hypothesis of Theorem* 5.1 *be true for the data of boundary value problem* (2.1). *Then, under condition* (4.5), *the solutions of the alternating difference schemes* (6.), (4.7), (3.3) *(or schemes* (6.), (4.7), (3.1)*) converges, as* N, $N_0 \to \infty$, *to the solution of the boundary value problem* ε*-uniformly (for a fixed value of the parameter). For the solutions of the difference schemes the estimates* (5.5) *hold.*

The similar finite difference constructions can be used to develop the parallel domain decomposition scheme with high-order accuracy in time.

Conclusion

In this paper we have constructed the parallel defect correction procedure that can easily be implemented in order to improve the time-accuracy, still obtaining ε-uniform second-order accuracy in the space discretization, as well as to parallelize computational performance of the finite difference schemes for a parabolic PDE.

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