International Conference on

Distributed Systems: Optimization and Economic-Environmental Applications

(DSO'2000)

Proceedings

Editors:

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Ekaterinburg, Russia,
30 May – 2 June, 2000

PARALLEL METHODS FOR QUASILINEAR SINGULARLY PERTURBED REACTION-DIFFUSION EQUATIONS [1]

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Keywords: elliptic PDEs, boundary layer, diffusion with predominant reaction, parallel computations, ε-uniform convergence.

On the rectangle $\overline{D}$, where $D = \{x : 0 < x_s < d_s, \ s = 1,2\}$ we consider Dirichlet's problem for the quasilinear singularly perturbed elliptic equation

$$L_{(1)}(u(x)) \equiv \varepsilon^2 \sum_{s=1,2} a_s(x) \frac{\partial^2}{\partial x_s^2} u(x) - g(x, u(x)) = 0, \quad x \in D, \tag{1a}$$

$$u(x) = \varphi(x), \quad x \in \Gamma. \tag{1b}$$

Here $\Gamma = \overline{D} \setminus D$, the functions $a_s(x)$, $g(x,u)$ and $\varphi(x)$ are sufficiently smooth, respectively, on the sets $\overline{D}$, $\overline{D} \times R$ and on the sides Γ_j, $\Gamma = \cup_j \Gamma_j$, $j = 1, \ldots, 4$, $\varphi(x)$ is continuous on Γ. Assume $0 < a_0 \le a_s(x) \le a^0$, $x \in \overline{D}$,

$$(\partial/\partial u)\, g(x,u) \ge g_0 > 0 \quad \text{for all} \quad (x,u) \in \overline{D} \times \left[-M_{(2)}, M_{(2)}\right], \tag{2}$$

where $M_{(2)}$ is a sufficiently large number. The perturbation parameter ε takes arbitrary values from the half-interval $(0,1]$; say $\varepsilon \ll 1$.

Model problems of similar kind arise, for example, in numerical modelling of stationary diffusion processes accompanied by first-order chemical reactions. The parameter ε characterizes the diffusion coefficient of the involved matter, and the constant g_0 refer to the reaction rate.

As $\varepsilon \to 0$, regular boundary layers appear in small neighbourhoods of the smooth parts of the boundary Γ, and corner (elliptic) layers appear in a neighbourhood of the set Γ_0 of the corner points. The data in (1) are assumed to satisfy the necessary compatibility conditions on Γ_0.

For problem (1) we construct an iterative domain decomposition scheme, that converges ε-uniformly and also allows for parallel computations.

Let us first give an iteration-free difference scheme. On the set $\overline{D}$ we introduce the rectangular grid

$$\overline{D}_h = \overline{\omega}_1 \times \overline{\omega}_2, \tag{3}$$

where $\overline{\omega}_s = \left\{x_s^i : \ 0 = x_s^0 < \ldots < x_s^{N_s} = d_s\right\}$ is a (possibly) nonuniform mesh

[1] This work was supported in part by the Russian Foundation for Basic Research (grant 98-01-00362), and by the Netherlands Organization for Scientific Research NWO (dossiernr. 047-008-007).

on $[0, d_s]$, $s = 1, 2$. Define $h_s^i = x_s^{i+1} - x_s^i$, $x_s^i, x_s^{i+1} \in \overline{\omega}_s$; let $h \le MN^{-1}$, where[2] $h = \max_{i,s} h_s^i$, $N = \min[N_1, N_2]$.

We approximate problem (1) by the finite difference scheme

$$\Lambda_{(4)}(z(x)) \equiv \varepsilon^2 \sum_{s=1,2} a_s(x)\delta_{\overline{xs}\,\widehat{xs}}\, z(x) - g(x, z(x)) = 0, \quad x \in D_h, \tag{4a}$$

$$z(x) = \varphi(x), \quad x \in \Gamma_h. \tag{4b}$$

Here $D_h = D \cap \overline{D}_h$, $\Gamma_h = \Gamma \cap \overline{D}_h$, $\delta_{\overline{xs}\,\widehat{xs}} z(x) = z_{\overline{xs}\,\widehat{xs}}(x)$ is the second-order (central) difference derivative on the non-uniform mesh $\overline{\omega}_s$ [1].

The difference scheme (4), (3) is monotone [1]. Using majorant functions and taking into account certain *a priori* estimates (see, e.g., [2]), we find: $|u(x) - z(x)| \le M\varepsilon^{-1}N^{-1}$, $x \in \overline{D}_h$. So this scheme converges, as $N \to \infty$, for fixed values of the parameter ε, but it does not converge ε-uniformly.

Note that we evaluate the errors in the maximum norm (in L_∞). Other norms, e.g., the L_p-norms $(1 \le p < \infty)$ and the weighted energy norm $\|\cdot\|_\varepsilon$, are unsatisfactory: in these norms the boundary layer vanishes for $\varepsilon \to 0$.

Now we introduce the special grid depending on ε

$$\overline{D}_h^* = \overline{\omega}_1^* \times \overline{\omega}_2^*. \tag{5}$$

Here $\overline{\omega}_s^* = \overline{\omega}_s^*(\sigma_s)$ is a piecewise uniform mesh refined in the neighbourhood of the end-points of $[0, d_s]$. In each interval $[0, \sigma_s]$ and $[d_s - \sigma_s, d_s]$ we use a fine mesh with step-size $h_s^{(1)} = 4\sigma_s N_s^{-1}$, and in $[\sigma_s, d_s - \sigma_s]$ we use a coarse mesh with step-size $h_s^{(2)} = 2(d_s - 2\sigma_s)N_s^{-1}$. We take $\sigma_s = \min[4^{-1}d_s, m_1^{-1}\varepsilon \ln N_s]$, where $0 < m_1 < m_0$, $m_0 = (g_0/a^0)^{1/2}$.

Let the solution $U_0(x)$ of the reduced equation $g(x, U_0(x)) = 0$, $x \in \overline{D}$, satisfy the estimate

$$\left| \partial^k U_0(x) / \partial x_1^{k_1} \partial x_2^{k_2} \right| \le M_{(6)}, \quad x \in \overline{D}, \quad k \le 2. \tag{6}$$

Theorem 1. *Assume in* (1) *that* $a_1, a_2 \in C^l(\overline{D})$, $g \in C^{l,l}(\overline{D} \times [-M_1, M_1])$, $\varphi \in C^l(\overline{D})$, *and let* $u \in C^l(\overline{D})$, $l = 4 + \alpha$, $\alpha > 0$, *where* $M_1 = M_0 M_{(6)}$, $M_0 > 18$, *and also let condition* (2) *be true with* $M_{(2)} = M_1$. *Then the difference scheme* (4), (5) *converges, as* $N \to \infty$, ε*-uniformly with an error bound given by*

$$|u(x) - z(x)| \le MN^{-2}\ln^2 N, \quad x \in \overline{D}_h^*.$$

Let the connected sets

$$D^k, \quad k = 1, \ldots, K \tag{7}$$

with piecewise smooth boundaries Γ^k, $\Gamma^k = \overline{D}^k \setminus D^k$, cover the domain D:

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[2] Here and below M, M_i (m, m_i) denote sufficiently large (small) positive constants independent of ε.

$D = \bigcup_{k=1}^{k} D^k$. We denote by $D^{[k]}$ the union of the subdomains $D^1, \dots, D^K$ excluding the set D^k: $D^{[k]} = \bigcup_{i=1,\ i \neq k}^{K} D^i$. We denote the minimal overlap of the sets D^k and $D^{[k]}$ by Δ^k, and also let Δ denote the least value of Δ^k, $k = 1, \dots, K$, i.e.

$$\Delta = \min_{k,\, x^1,\, x^2} \rho(x^1, x^2), \quad x^1 \in \overline{D}^k, \ x^2 \in \overline{D}^{[k]}, \quad x^1, x^2 \notin \{ D^k \cap D^{[k]} \}, \tag{8}$$

$k = 1, \dots, K$, where $\rho(x^1, x^2)$ is the distance between the points $x^1, x^2 \in \overline{D}$. In general, the value Δ may depend on the parameter ε: $\Delta = \Delta(\varepsilon)$.

Now we construct a Schwartz method modified for implementation in a parallel environment with $P \geq 1$ processors. This construction follows [3,4].

Let each subdomain D^k, $k = 1, \dots, K$ from (7) be partitioned in P disjoint non-overlapping parts (some of them may be empty): $D^k = \bigcup_{p=1}^{P} D_p^k$, $k = 1, \dots, K$, $\overline{D}_i^k \cap \overline{D}_j^k = \emptyset$, $i \neq j$. Assume $\Gamma_p^k = \overline{D}_p^k \setminus D_p^k$.

On the sets $\overline{D}_p^k$ we construct the coherent meshes

$$\overline{D}_{ph}^k = \overline{D}_p^k \cap \overline{D}_h, \quad k = 1, \dots, K, \quad p = 1, \dots, P, \tag{9a}$$

where $\overline{D}_h = \overline{D}_{h(3)}$ or $\overline{D}_h = \overline{D}^{*}_{h(5)}$. Let $\Gamma_{ph}^k = \overline{D}_{ph}^k \setminus D_p^k$.

Given the function $z^0(x)$ on $\overline{D}_h$ satisfying condition (4b), we find the sequence of the functions $z^r(x)$, $x \in \overline{D}_h$, $r = 1, 2, \dots$ from the solutions of the discrete problems

$$\left. \begin{aligned} \Lambda_{(4)}\left(z_p^{r+\frac{k}{K}}(x)\right) &= 0, & x &\in D_{ph}^k, \\ z_p^{r+\frac{k}{K}}(x) &= z^{r+\frac{k-1}{K}}(x), & x &\in \Gamma_{ph}^k \end{aligned} \right\}, \quad p = 1, \dots, P; \tag{9b}$$

$$z^{r+\frac{k}{K}}(x) = \left\{ \begin{aligned} & z_p^{r+\frac{k}{K}}(x), \quad x \in \overline{D}_{ph}^k, \ p = 1, \dots, P, \\ & z^{r+\frac{k-1}{K}}(x), \quad x \in \overline{D}_h \setminus \overline{D}^k \end{aligned} \right\}, \quad \begin{aligned} & x \in \overline{D}_h, \\ & k = 1, \dots, K, \end{aligned}$$

$$z^{r+1}(x) = z^{r+\frac{K}{K}}(x), \quad x \in \overline{D}_h; \qquad r = 0, 1, 2, \dots . \tag{9c}$$

Under the condition

$$\Delta = \Delta_{(8)}(\varepsilon) > 0, \ \ \varepsilon \in (0,1], \quad \inf_{\varepsilon \in (0,1]} \left[\varepsilon^{-1} \Delta_{(8)}(\varepsilon) \right] > 0, \tag{10}$$

we obtain the estimate (see [3] for the linear case)

$$| z(x) - z^r(x) | \leq M q^r, \quad x \in \overline{D}_h, \tag{11}$$

where $q \leq 1 - m$, e. g., we can take $q = \exp(-m_0\varepsilon^{-1}\Delta)$. If condition (10) is violated, the functions $z^r(x)$ do not converge ε-uniformly with respect to the number r of iterations. On the special grid $\overline{D}^{\star}_{h(5)}$ we have the estimate

$$|\,u(x) - z^r(x)\,| \leq M\left[\,N^{-2}\ln^2 N + q^r\,\right], \quad x \in \overline{D}^{\star}_h. \tag{12}$$

Emphasize that the estimate for q in (11), (12) is independent of ε.

Theorem 2. *The condition* (10) *is necessary and sufficient for ε-uniform convergence (as $r \to \infty$) of the functions $z^r(x)$, i. e., the solutions of the decomposition scheme* (9), (3) *with P parallel solvers, to the solution $z(x)$ of the base scheme* (4), (3). *If the hypotheses of Theorem* 1 *and also condition* (10) *are fulfilled, the solutions $z^r(x)$ of scheme* (9), (5) *converge, as $N, r \to \infty$, to the solution $u(x)$ of the boundary value problem* (1) *ε-uniformly. Under condition* (10), *the estimates* (11), (12) *are valid.*

The difference scheme is nonlinear. To solve the problem, we apply a linearization procedure [1], replacing $g(x, z^r(x))$ by $c(x)(z^r_{(i)}(x) - z^r_{(i-1)}(x)) + g(x, z^r_{(i-1)}(x))$. Here $z^r_{(i)}(x)$ is the i-th inner iteration, the function $c(x)$ satisfies the condition $c(x)u - g(x,u) \geq c_0 u,\ c_0 > 0$, where $(\partial/\partial u)g(x,u) \leq g^0$, $(x,u) \in \overline{D} \times [\,-M_{(2)}, M_{(2)}\,]$.

Thus, for parallelization of the computational method, we have constructed iterative difference schèmes that converge ε-uniformly with respect to both the number N of grid nodes and the number r of (outer) iterations required for convergence of the iterative process.

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